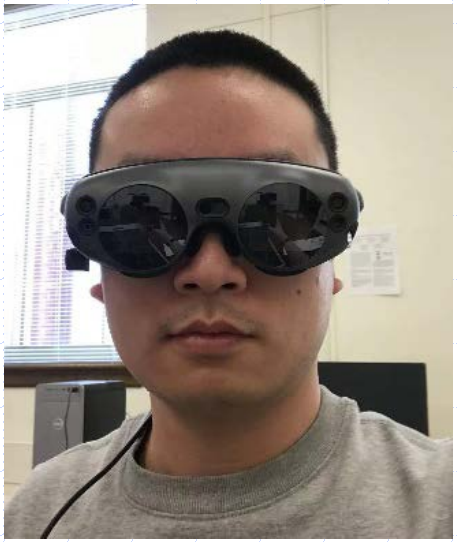
CS4140
Embedded Systems Laboratory

Introduction to Digital Filtering

Brief Intro



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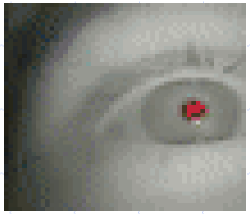
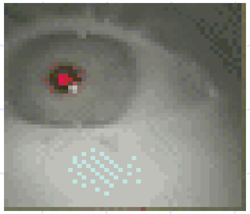
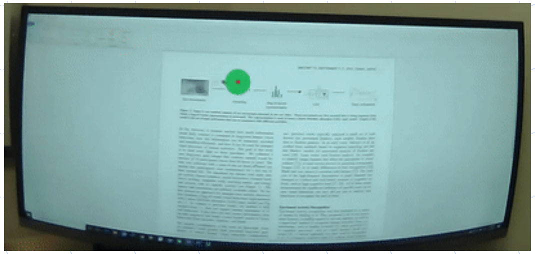
Things that I am working on







Advertisement:

Multiple openings for MSc project on (1) self-learning for eye tracking and gaze-based context sensing systems, (2) privacy-preserving AR/VR.

Reference

Steven W. Smith, "*The scientist and engineer's guide to digital signal processing*." 1997.

Sanjeev R. Kulkarni, "*Lecture Notes for ELE201 Introduction to Electrical Signals and Systems*", Princeton University, 2002.

Why Signal Processing?

- Improve/restore media content
  - Compression/Decompression
  - Audio filtering (bass, treble, equalization)
  - Video filtering (enhancement, contours, ..)
  - Noise suppression (accel, gyro data)
  - Data fusion (mixing accel + gyro data)
- By digital means: DSP

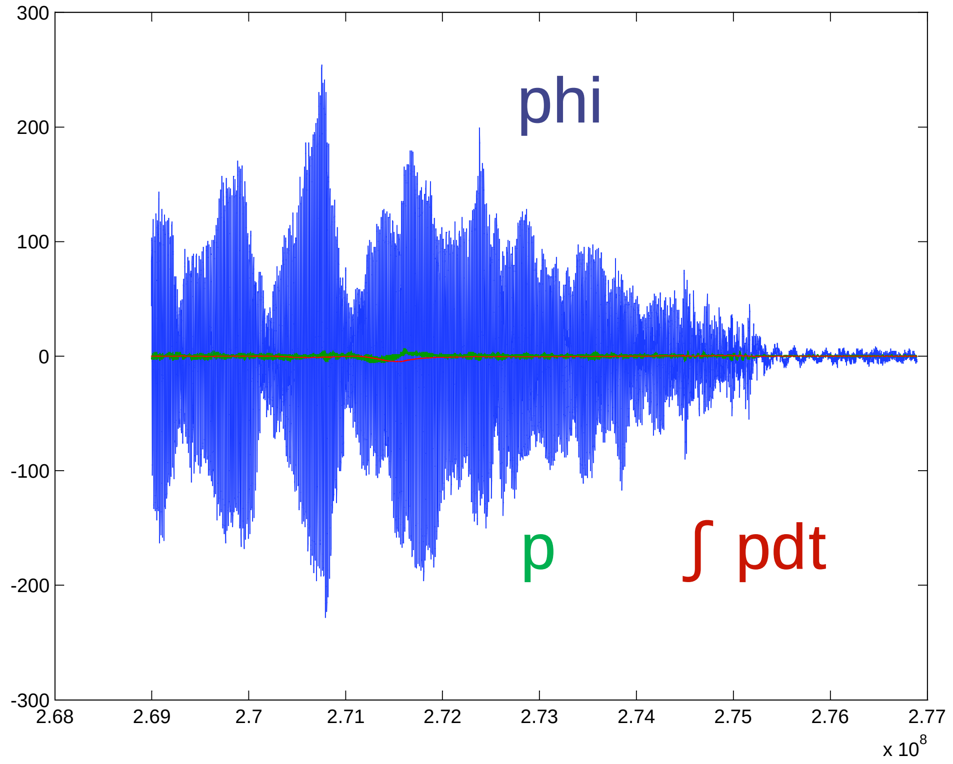
DSP is Everywhere

- Cell Phone
- TV
- Plant Control
- Climate Control
- Automotive
- Copiers, Wafer Scanners
- Model Quad Rotors …

Objectives of this Crash Course

- Appreciate the benefits of Digital Filtering
- Understand *some* of the basic principles
- Communicate with DSP engineers
- Implement your own filters for the QR

Example: QR Sensor Signals phi, p



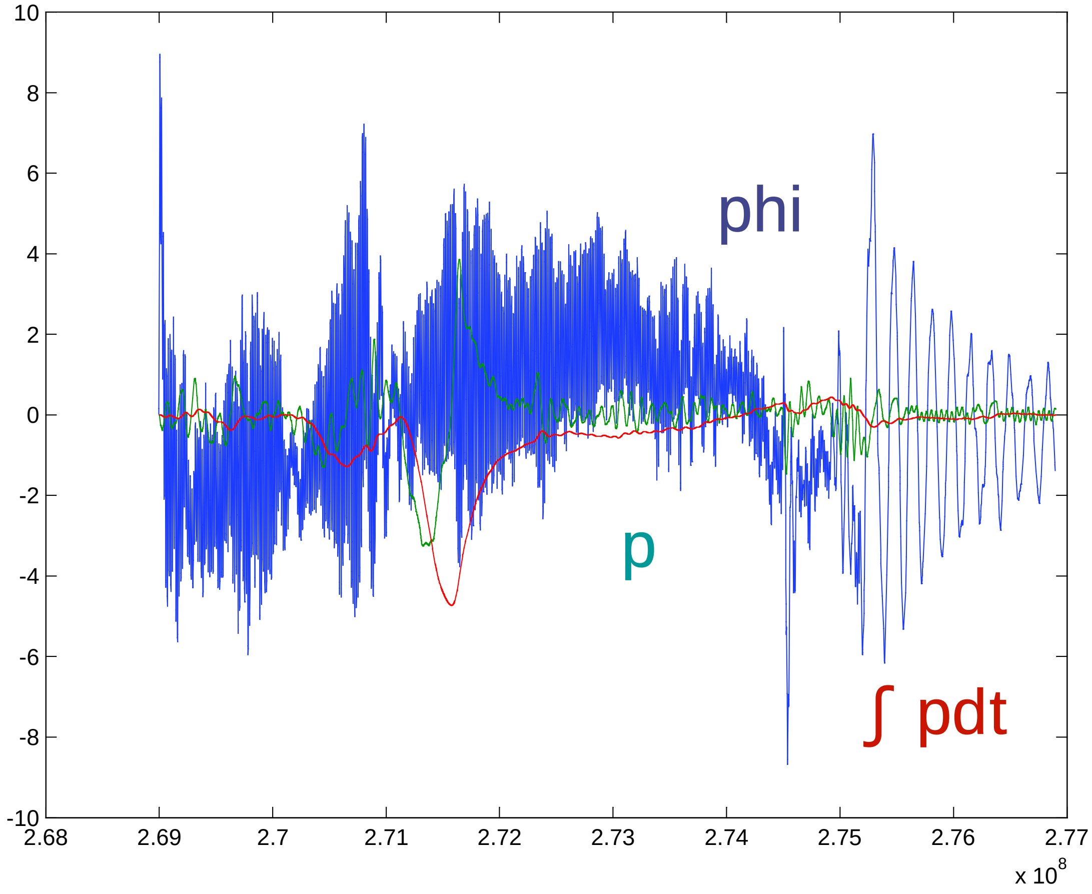
Signal from the previous version of QR:

- phi is the plot of Acc
    - Roll angle
- p is the roll rate
- red is the
    - Integral of roll rate = roll angle

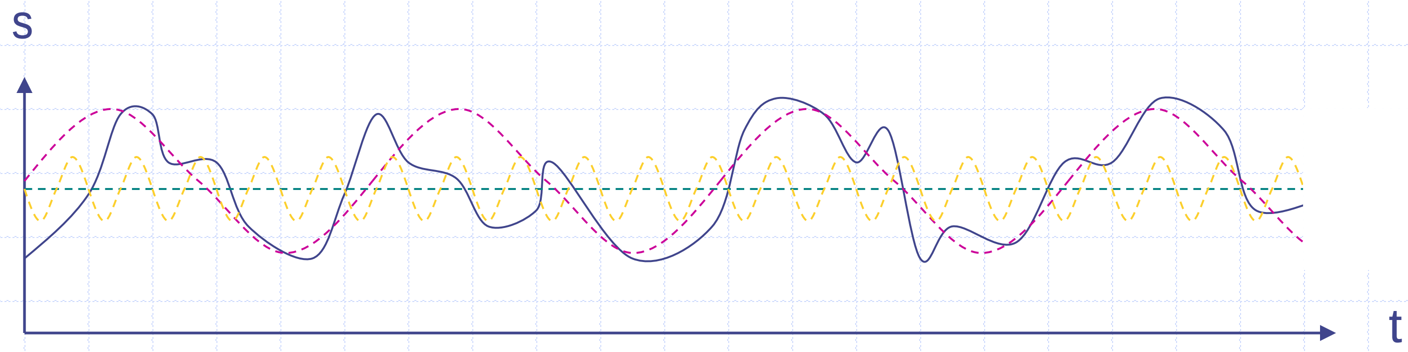
Issues?

- Scaling
- Noise

After some low-pass filtering



Signals and Frequency Synthesis

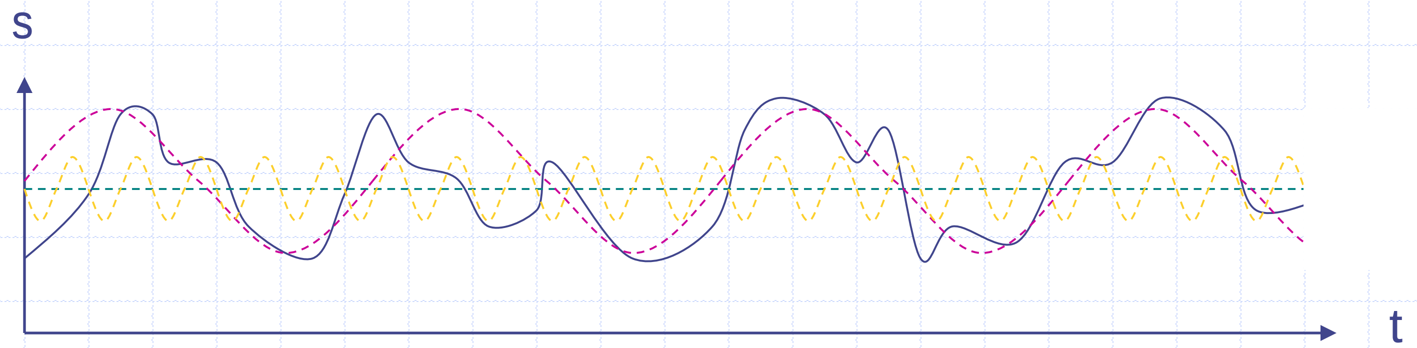


Usually signals (such as s) are composed of signals with many frequencies.
For instance, s contains

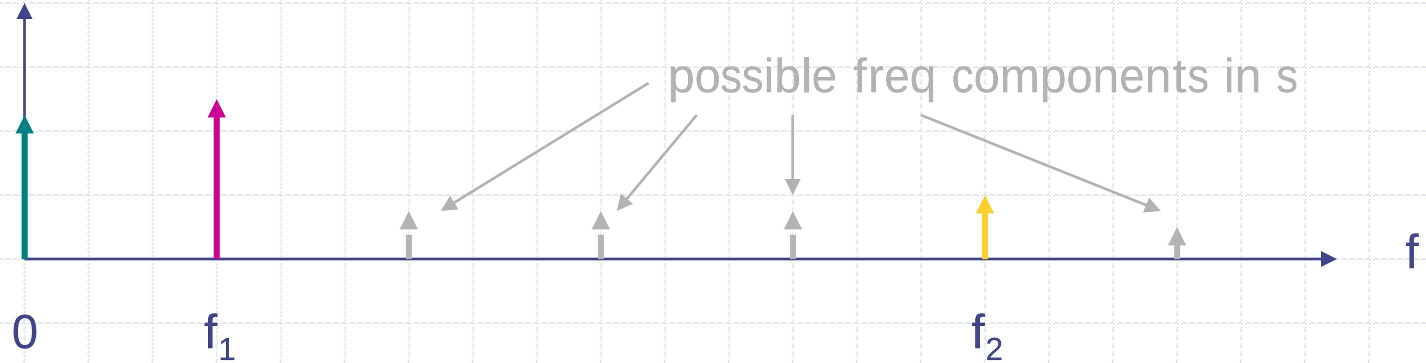
- 0 Hz component (green dashed line) ---- DC term
- lowest freq component (purple dashed line)
- higher freq component (yellow dashed line)
- and others

Fourier: Any *periodic* signal with base frequency f_b
can be constructed from sine waves with frequency f_b, $2f_b$, $3f_b$, ...

Frequency Spectrum

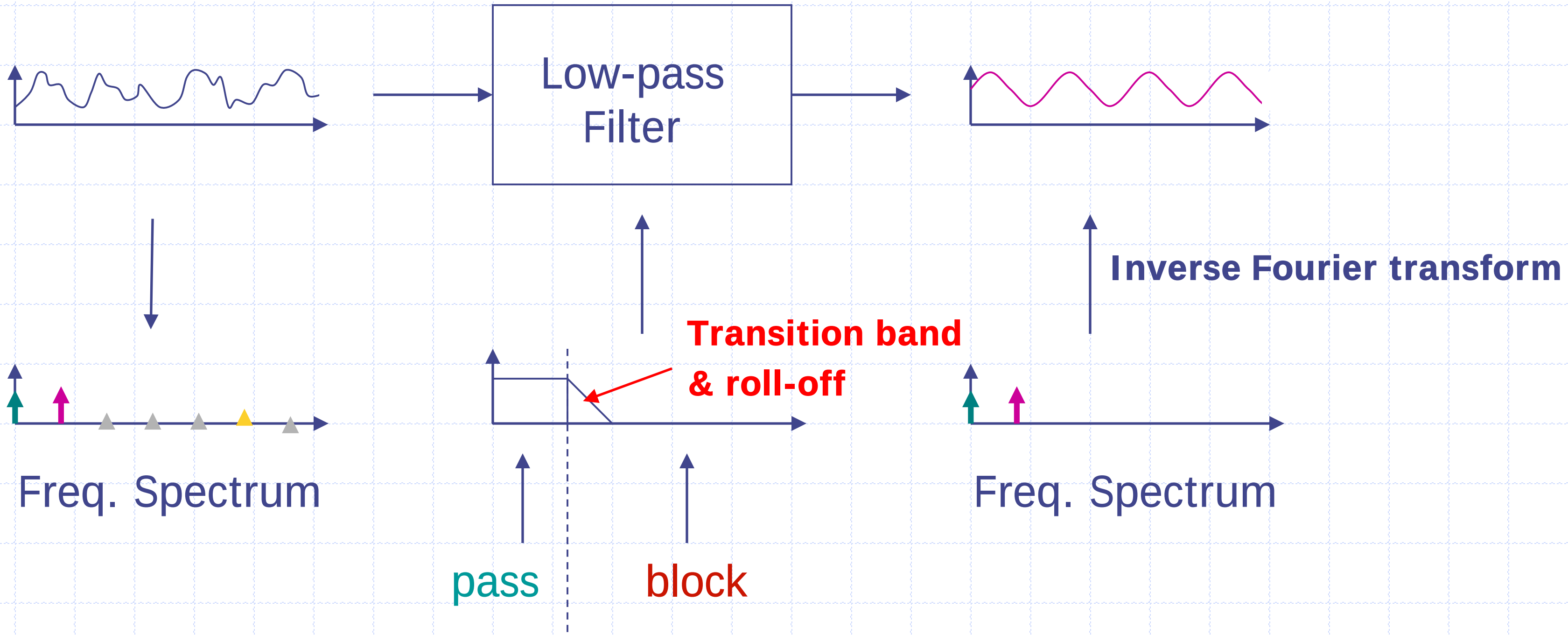


The frequency spectrum of s is:



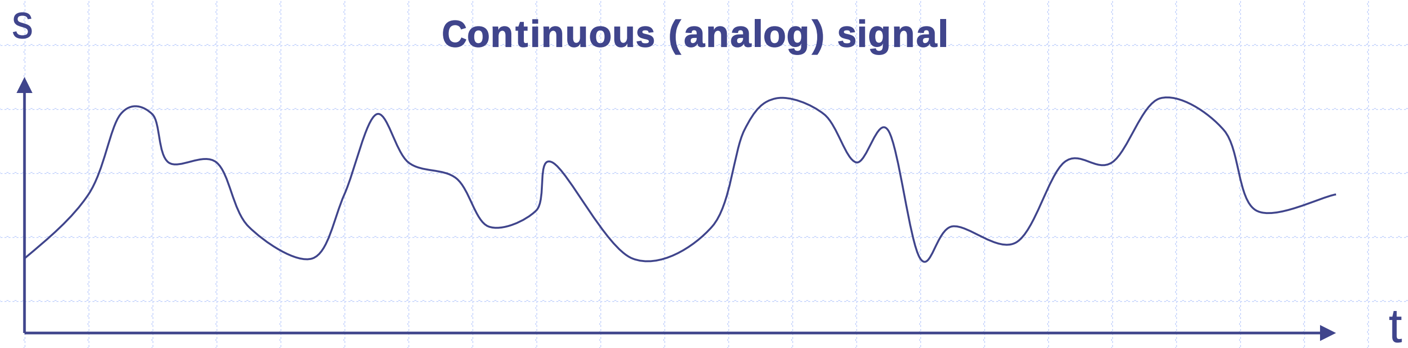
Filter: Frequency Response

Often filters are designed to filter frequency components in a signal

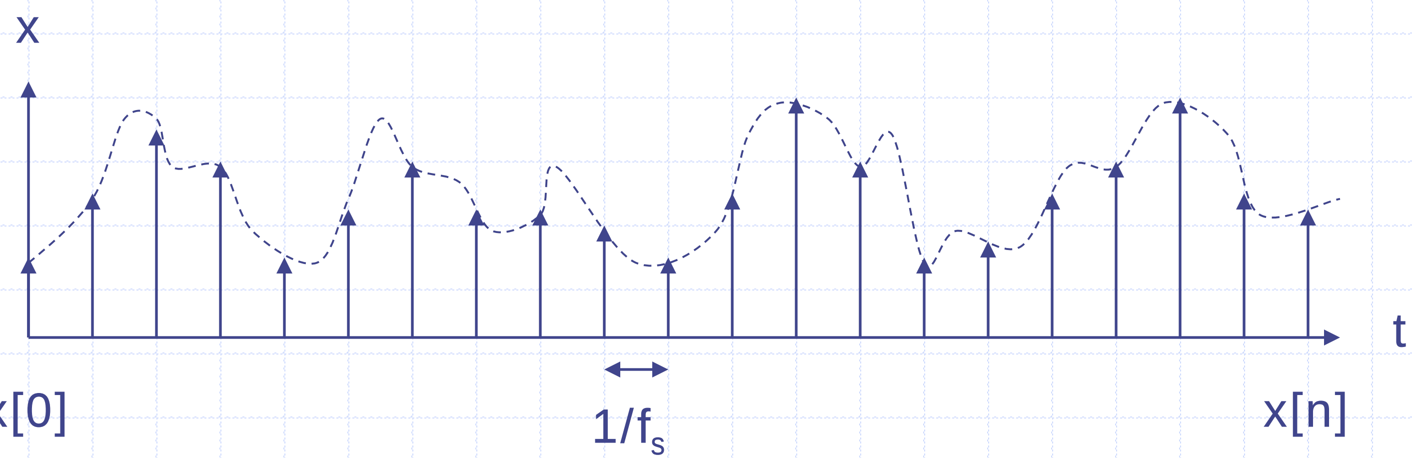


Filter's *Frequency Response*

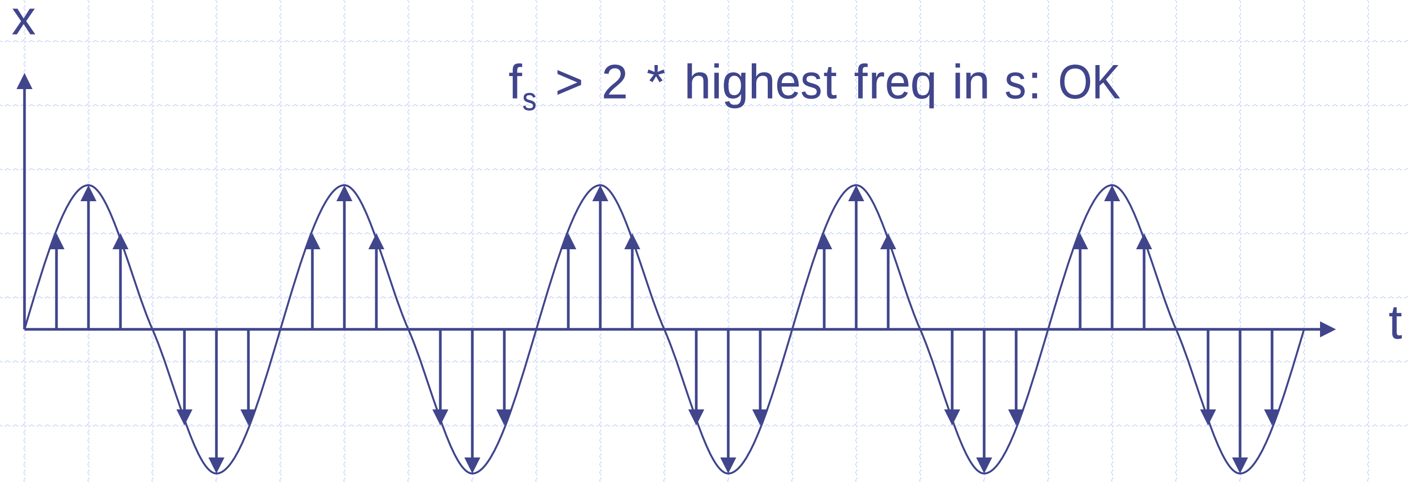
Sampling A Signal

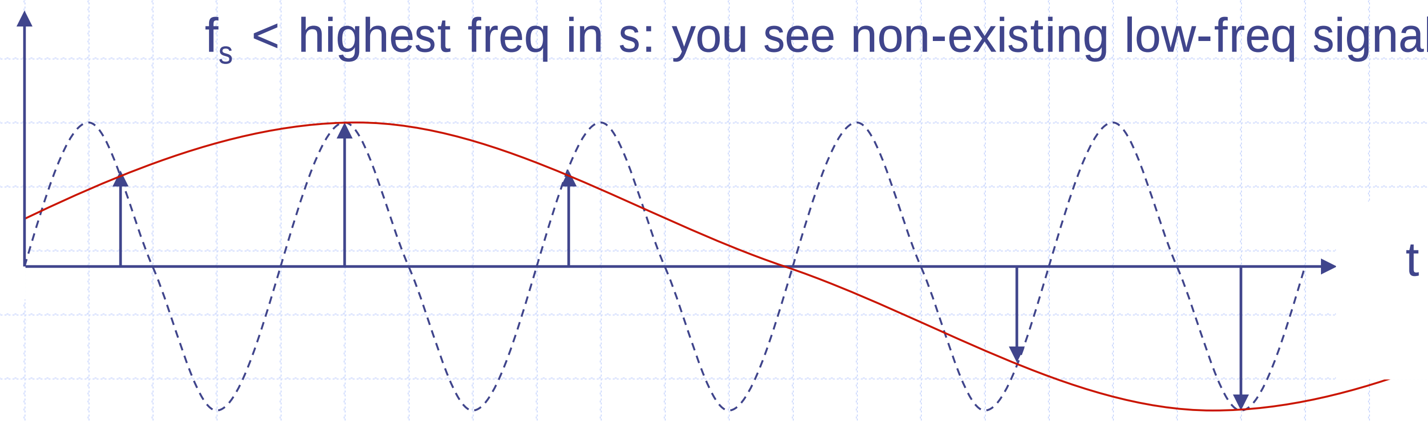


s sampled at *discrete* time intervals (sample frequency f_s): x[n]

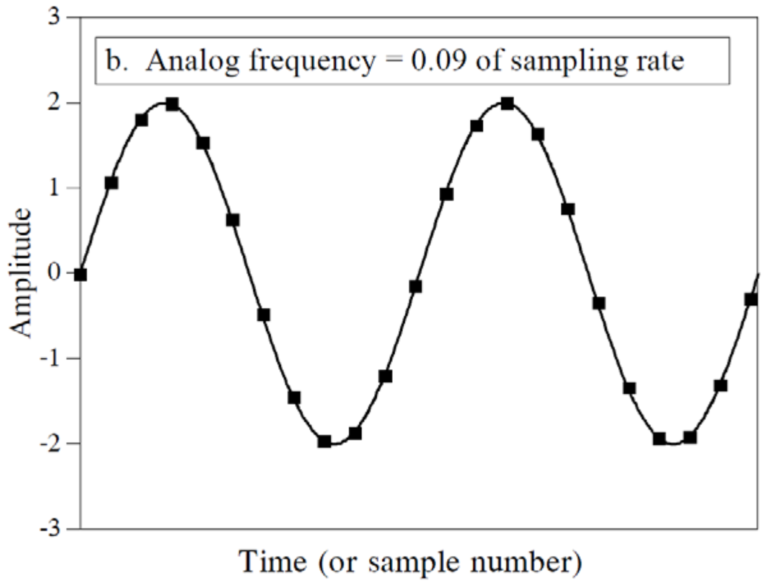


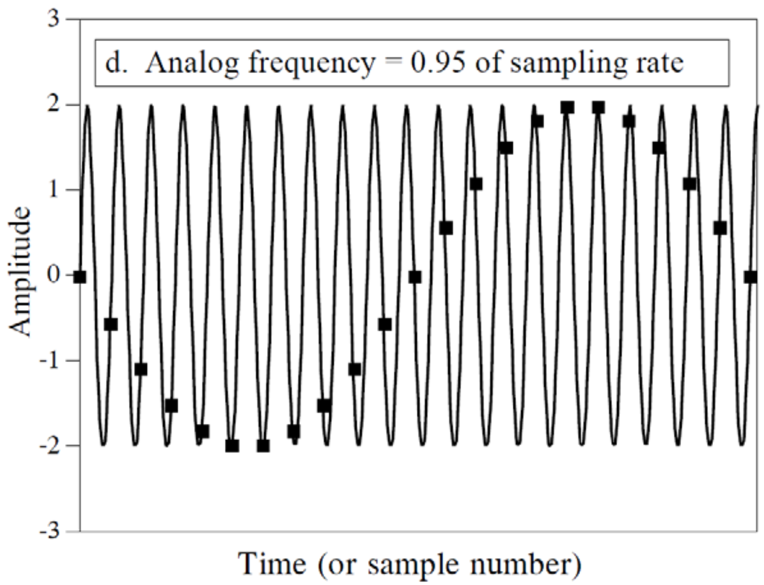
Sampling: Avoid Aliasing





Sampling: Avoid Aliasing (cont.)

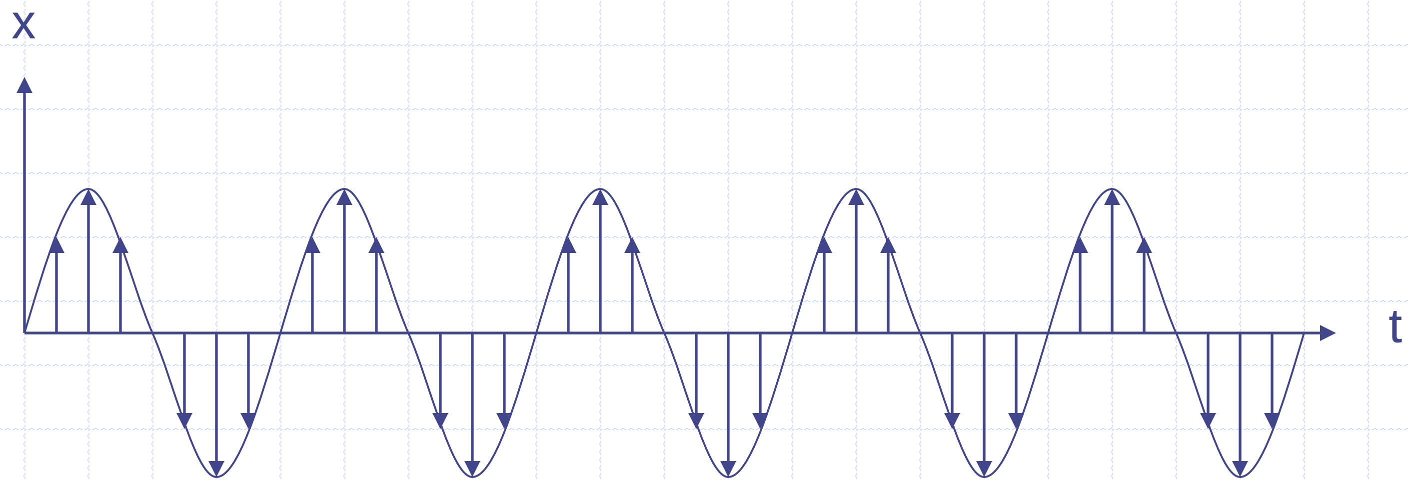




fs = 10 f

fs = f

Sampling: Avoid Aliasing (cont.)

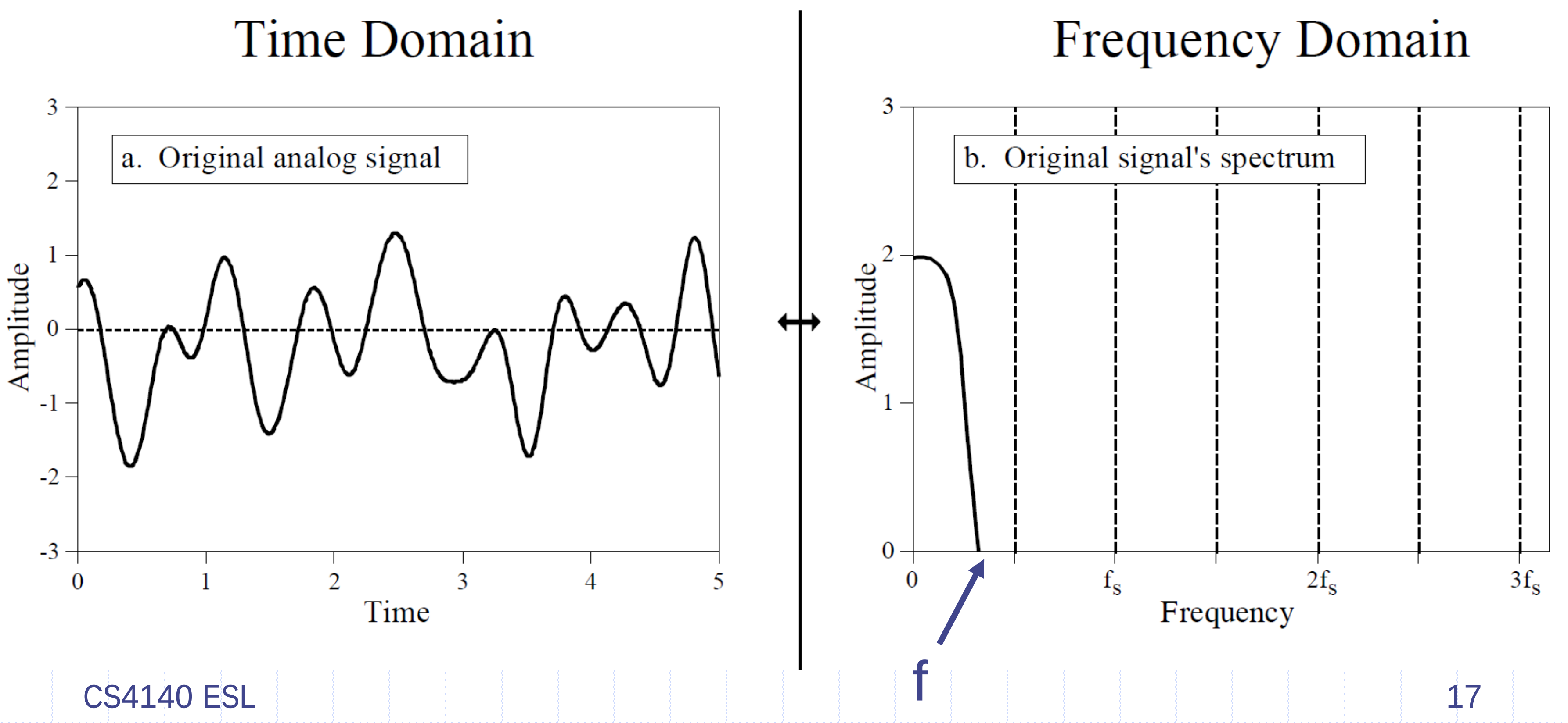


Shannon Sampling Theorem: a bandlimited signal with maximum frequency s can be perfectly reconstructed from samples if the sampling frequency satisfies:

$$f_s > 2 * \text{highest freq in } s$$

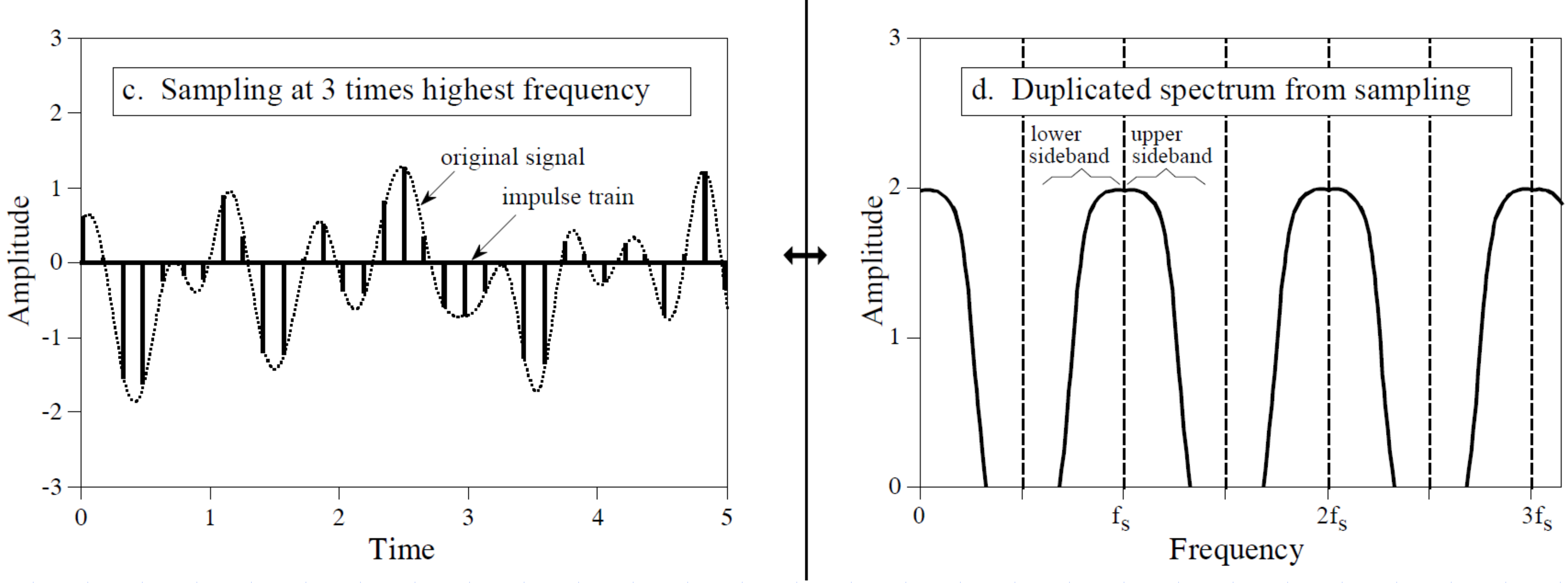
Sampling: Why Aliasing Happen

An analog signal composed of frequency components between 0 and f



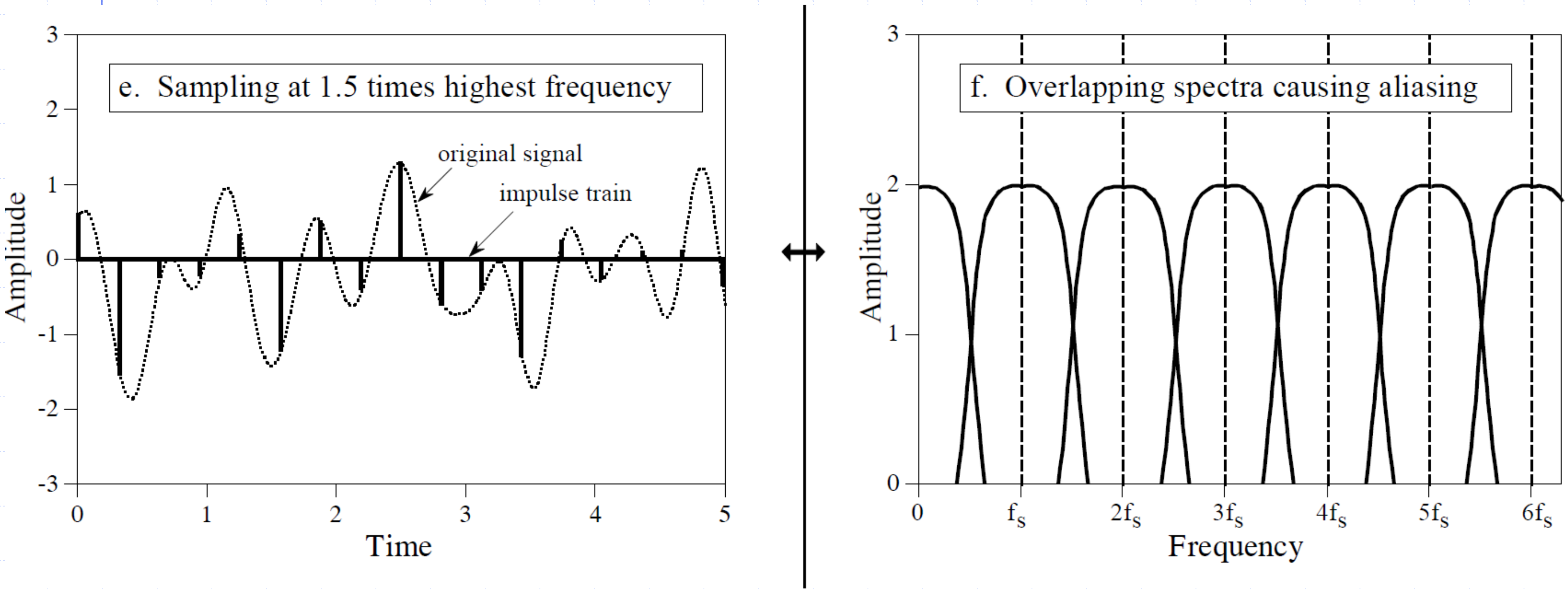
Sampling: Why Aliasing Happen (cont.)

Sampling the original signal using an impulse train.
The spectrum is then a duplication of the spectrum of the original at Multiple of the sampling frequency, i.e., fs, 2fs, 3fs, etc.



Sampling: Why Aliasing Happen (cont.)

Overlapping happened in the spectrum when fs<2f



Example Filter: Moving Average

y[n] = 1/3 x[n] + 1/3 x[n-1] + 1/3 x[n-2]

$$y[i] = \frac{1}{M}\sum_{j=0}^{M-1} x[i-j]$$

x[n] → MA Filter → y[n]

```
x[0] = get_sample();
y[0] = (x[0]+x[1]+x[2])/3;
put_sample(y[0]);
x[2] = x[1]; x[1] = x[0];
```

MA filter filters (removes) signals of certain frequency:

x, freq f, amplitude 1 → MA Filter → y, freq f, amplitude ???

Frequency Behavior MA

lower frequency x: amplitude y = 0.77, fs = 12 f

x = 0.00, 0.33, 0.66, 1.00, 0.66, 0.33, 0.00, -0.33, -0.66, -1.00, -0.66, -0.33, 0.00

y = 0.00, 0.11, 0.33, 0.66, 0.77, 0.66, 0.33, 0.00, -0.33, -0.66, -0.77, -0.66, -0.33

higher frequency x: amplitude y = 0.33, fs = 4 f

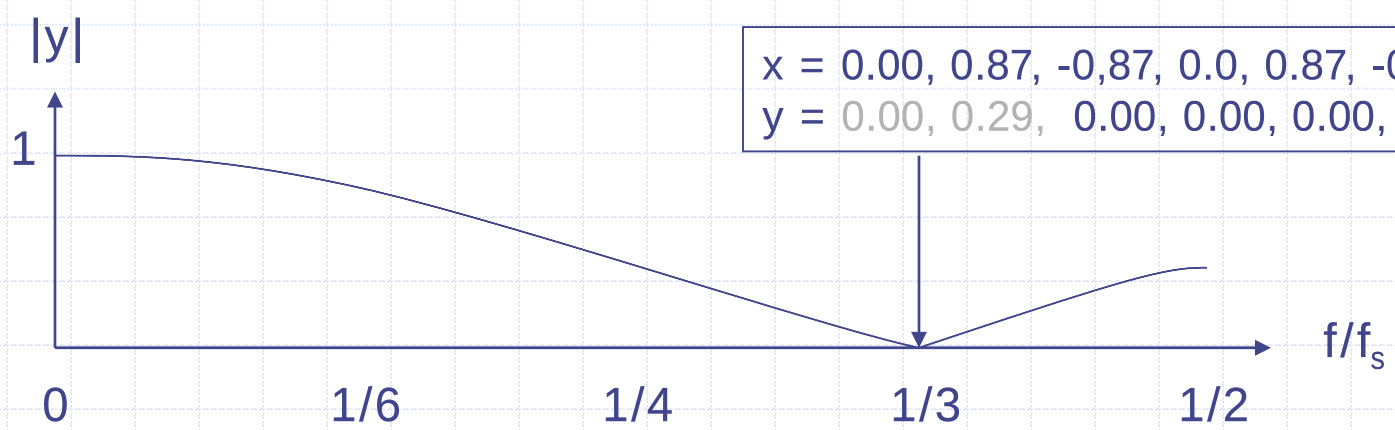
x = 0.00, 1.00, 0.00, -1.00, 0.00, 1.00, 0.00, -1.00, 0.00, 1.00, 0.00, -1.00, 0.00

y = 0.00, 0.33, 0.33, 0.00, -0.33, 0.00, 0.33, 0.00, -0.33, 0.00, 0.33, 0.00, -0.33

transient — steady-state

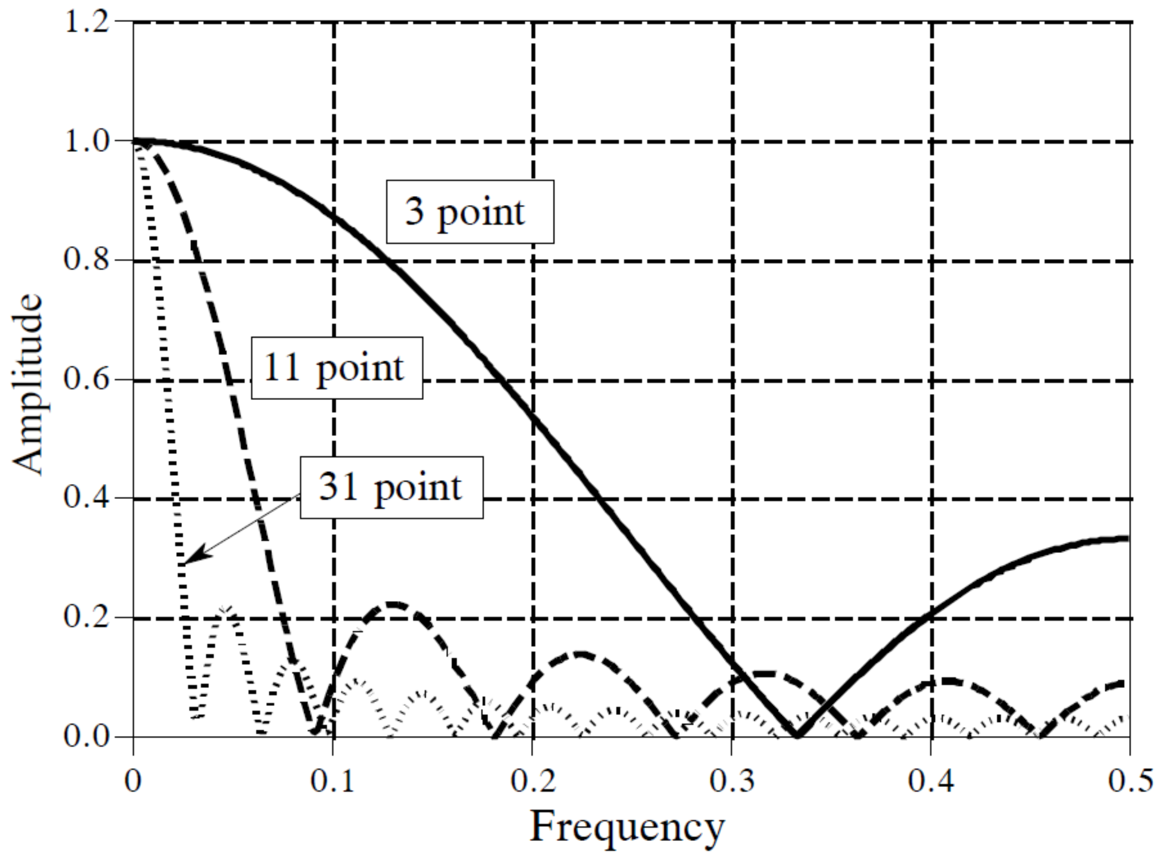
x = 0.00, 0.87, -0,87, 0.0, 0.87, -0.87, 0.00

y = 0.00, 0.29, 0.00, 0.00, 0.00, 0.00, 0.00



Frequency Behavior MA (cont.)

Frequency response of MA filter:
“X point” refers to the window size



Outline

- Introduction
- Z Transform
- FIR Filters
- IIR Filters
- Fixed-point Implementation
- Kalman Filter

Analysis: Z Transform

- We can numerically evaluate frequency behavior
- Rather analyze frequency behavior through *analytic* means
- For this we introduce Z transformation

- Let x[n] be a signal in the time domain (n)
- The Z transform of x[n] is given by

$$X(z) = \Sigma_n x[n] z^{-n}$$

where z is a complex variable.

- Example:

$x = 0.00, 0.33, 0.66, 1.00, 0.66, ..$

$X = 0 + 0.33z^{-1} + 0.66z^{-2} + z^{-3} + 0.66z^{-4} + ...$

Z Transform

- Z transforms make life easy
- Properties of the Z transform, Shifting:

- Let y[n] = x[n-1]   (i.e., signal delayed by 1 sample)

$$Y(z) = z^{-1} X(z)$$

- Example:

$x = 0.00, 0.33, 0.66, 1.00, 0.66, ..$

$X = 0 + 0.33z^{-1} + 0.66z^{-2} + z^{-3} + 0.66z^{-4} + ...$

$y = 0.00, 0.00, 0.33, 0.66, 1.00, ..$

$Y = 0 + 0z^{-1} + 0.33z^{-2} + 0.66z^{-3} + z^{-4} + ...$

$\quad = z^{-1} X$

Z Transform

- Other properties of the Z transform:

- Z transform of K a[n] = K A(z)
- Z transform of a[n] + b[n] = A(z) + B(z)

- Example:

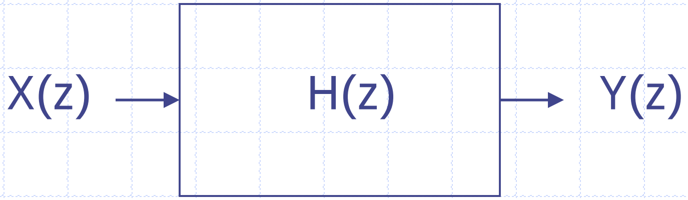
$$x = 0.00, 0.33, 0.66, 1.00, 0.66, ..$$
$$X = 0 + 0.33z^{-1} + 0.66z^{-2} + z^{-3} + 0.66z^{-4} + ...$$
$$y = 0.00, 0.66, 1.32, 2.00, 1.32, ..$$
$$Y = 0 + 0.66z^{-1} + 1.32z^{-2} + 2.00z^{-3} + 1.32z^{-4} + ...$$
$$= 2X$$

Apply Z transform to MA Filter

y[n] = 1/3 x[n] + 1/3 x[n-1] + 1/3 x[n-2]

In terms of the Z transform we have:

$$
\begin{aligned}
Y(z) &= 1/3\, X(z) + 1/3\, z^{-1} X(z) + 1/3\, z^{-2} X(z) \\
&= (1/3 + 1/3\, z^{-1} + 1/3\, z^{-2})\, X(z) \\
&= H(z)\, X(z)
\end{aligned}
$$



- It holds Y(z) = H(z) X(z), where H(z) is filter's (system's) transfer function
- Frequency response of filter can be read from H(z)

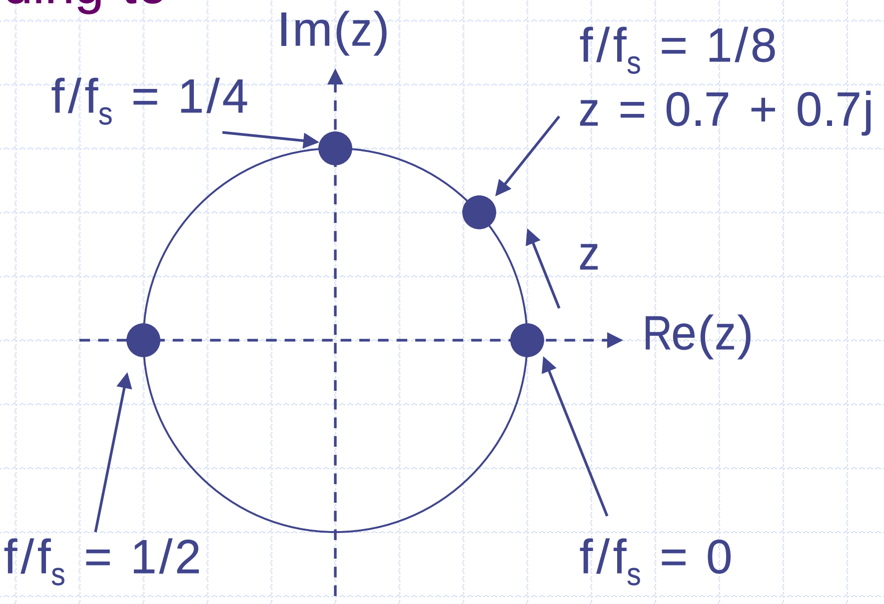
Frequency Response H(z)

H(z) reveals frequency response ($H(f)=H(z)|\ z=e^{j2\pi f}$):
As $Y(z) = H(z)\ X(z)$, $|H(z)|$ determines *amplification* of X(z)

The variable z is a complex variable and encodes frequency $F= f/f_s$ according to

$$z = e^{j2\pi F}$$
$$= \cos(2\pi F) + j \sin(2\pi F)$$

This corresponds to traversing the unit circle in the complex z plane:



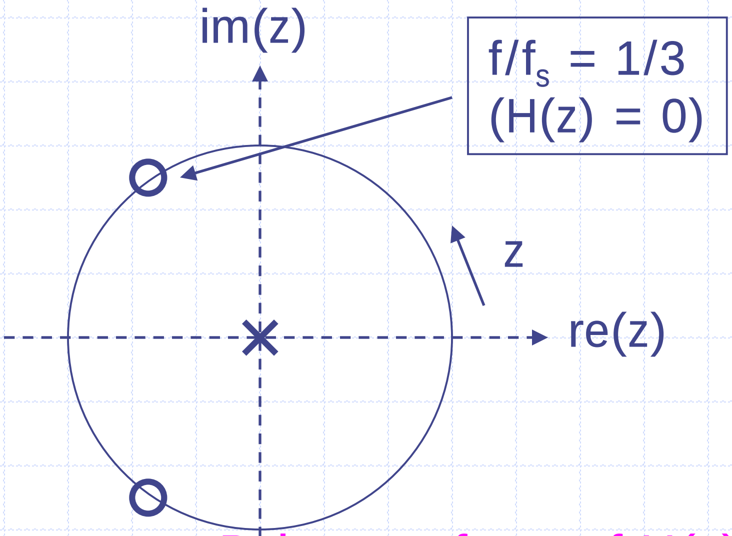
Frequency Response MA Filter

The transfer function of the MA filter is given by:

$H(z) = (1/3 + 1/3\ z^{-1} + 1/3\ z^{-2})$
$\quad = (1/3\ z^2 + 1/3\ z + 1/3) / z^2$ (normalized)

Determine poles and zeros of H(z):

zero (= root of numerator):
$z_1 = -½+½\sqrt{3}j$, $z_2 = -½-½\sqrt{3}j$
$(H(z_{1,2}) = 0)$
pole (= root of denominator):
$z_3, z_4 = 0$
$(H(z_{3,4}) = \infty)$



Simply inspect distance z to poles/zeros.

Pole-zero form of H(z):

$$H[z] = \frac{(z - z_1)(z - z_2)(z - z_3)\cdots}{(z - p_1)(z - p_2)(z - p_3)\cdots}$$

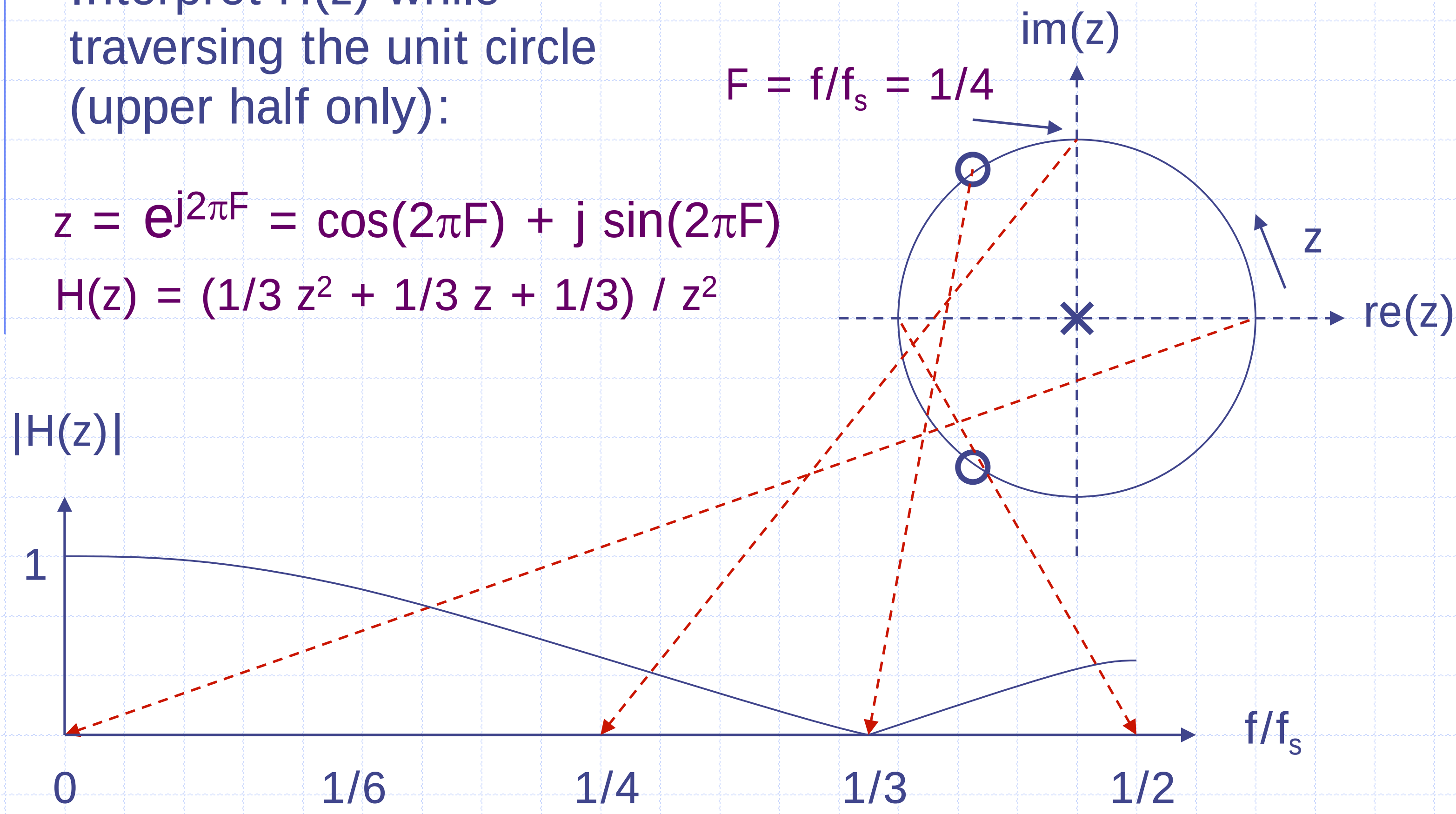
Frequency Response MA Filter

A practice!

Interpret H(z) while traversing the unit circle (upper half only):

$$z = e^{j2\pi F} = \cos(2\pi F) + j\sin(2\pi F)$$

$$H(z) = (1/3\, z^2 + 1/3\, z + 1/3) / z^2$$



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Impulse Response

Impulse signal $\delta[n]$ = 1, 0, 0, 0, ... (a spike, Dirac pulse)

Impulse response (IR) of a filter:

$\delta[n] \longrightarrow \boxed{\ H\ } \longrightarrow y[n]$, characteristic for H

MA filter: $y[n] = 1/3\ x[n] + 1/3\ x[n-1] + 1/3\ x[n-2]$

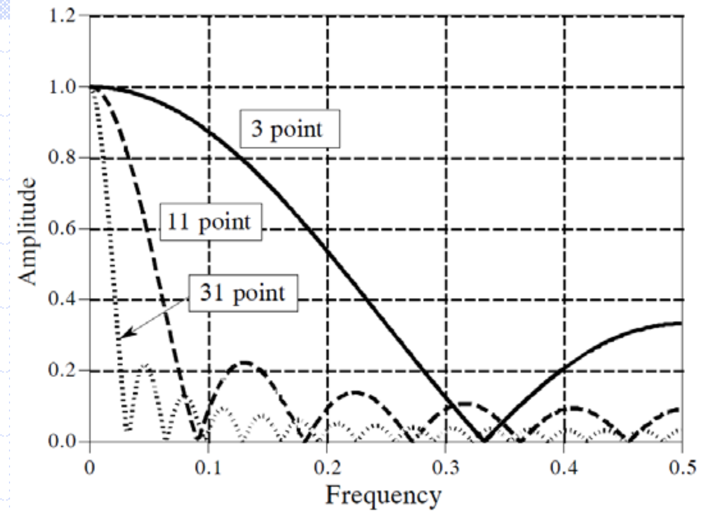
Let $x[n] = \delta[n]$, then $y[n]$ = 1/3, 1/3, 1/3, 0, 0, 0, ...

The transfer function

Z Transform: $X(z) = 1$, $Y(z) = H(z) * 1 \Rightarrow \boxed{H(z) = 1/3 + 1/3z^{-1} + 1/3z^{-2}}$

Impulse signal δ reveals $H(z)$ in terms of $h[n]$

Impulse Response



MA filter: h[n] = 1/3, 1/3, 1/3, 0, 0, 0, ...

The IR is finite: as it settles to zero in finite time.

Filters defined by

$y[n] = a_0\, x[n] + a_1\, x[n-1] + a_2\, x[n-2] + \ldots$

The output is a discrete convolution of the input signal and the IR.

Always have a finite IR and are therefore called FIR filters
(the equation is non-recursive in y)

Although any filter can be designed, FIR filters are
costly in terms of computation (often many terms needed)

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Averaging Filter

Suppose we want to extend MA filter to N terms:

y[n] = 1/N x[n] + 1/N x[n-1] + ... 1/N x[n-N-1]

Suppose we don't want to implement an N-cell FIFO + 2N ops and experiment with the following "short cut":

y[n] = ((N-1)/N) * y[n-1] + 1/N * x[n]

(1st term approximates contents of FIFO after x[n-N-1] has been shifted out, 2nd term is newest sample shifted in)

Let's analyze the frequency response of this filter (recursive filter)

Frequency Response Filter

$y[n] = (N-1)/N \; y[n-1] + 1/N \; x[n]$

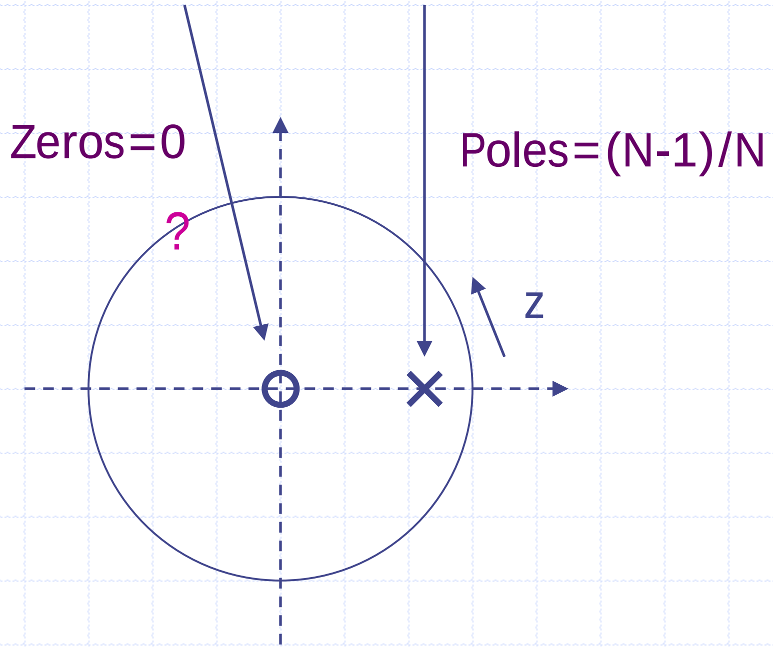
$Y(z) = (N-1)/N \; z^{-1} \; Y(z) + 1/N \; X(z)$

$H(z) = (1/N) \;/\; (1 - (N-1)/N \; z^{-1})$

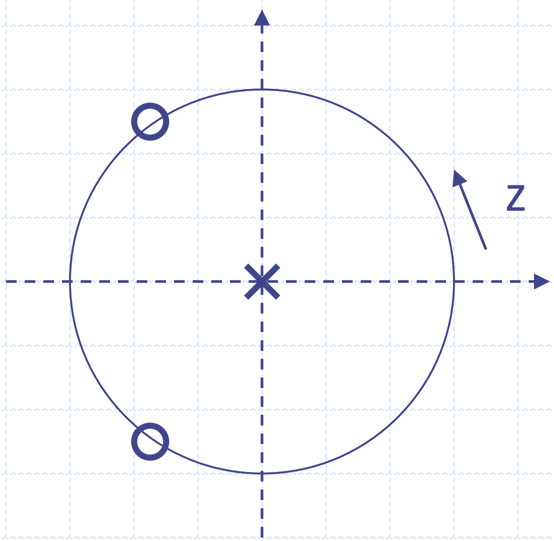
$\quad\;\; = (z/N) \;/\; (z - (N-1)/N)$

Z-transform

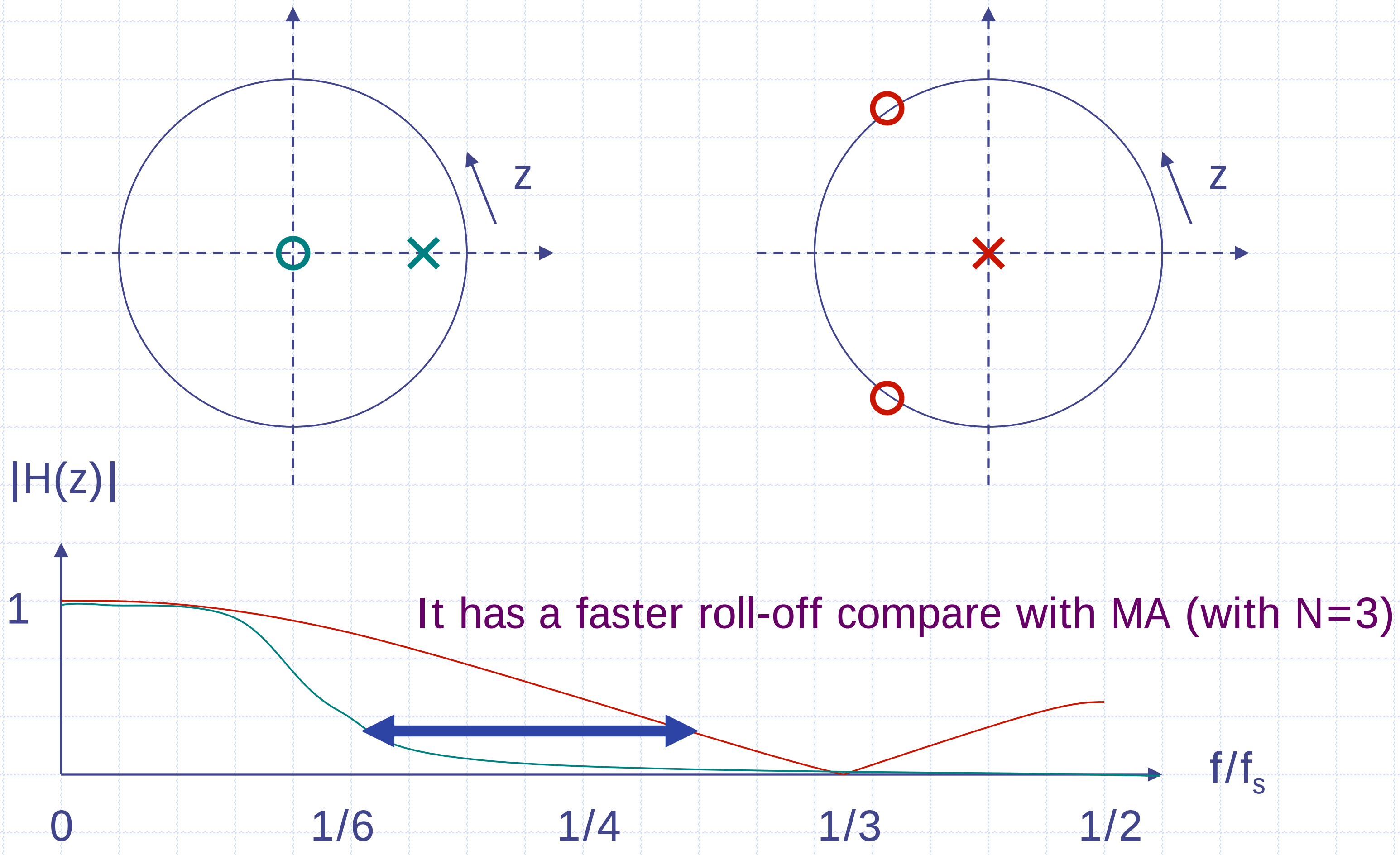
$H(z) = Y(z)/X(z)$



cf. MA filter:



Frequency Response Comparison



Comparison of both Filters

New filter is much more different than perhaps assumed

Pole-zero plot is quite different:
now poles not zero: play an active role

Frequency response is (therefore) more low-pass than MA filter

The closer the pole is to unit circle (larger N),
the sooner is the cut-off (in terms of frequency f),
this generally corresponds to MA filter but this would take large FIFO!

Impulse Response

Filter equation: $y[n] = (N-1)/N \, y[n-1] + 1/N \, x[n]$

IR (N = 3): $h[n] = 1/3, (2/3)^1/3, (2/3)^2/3, \ldots, (2/3)^n/3, \ldots$

The IR is infinite: amplitude decays exponentially in n

Filters defined by

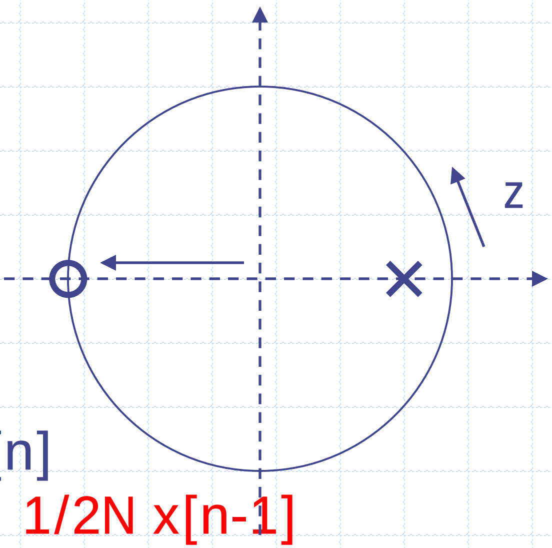
$b_0 \, y[n] + b_1 \, y[n-1] + \ldots = a_0 \, x[n] + a_1 \, x[n-1] + \ldots$

always have an infinite IR and are therefore called IIR filters (the equation is recursive in y)

Filter order determined by # coefficients. Our case: 1st order.

Designing Filters

Looking at the pole-zero plot, the IIR filter can be improved by moving zero to left:
now |H(z)| even becomes zero for f = $f_s/2$
so sharper cut-off.



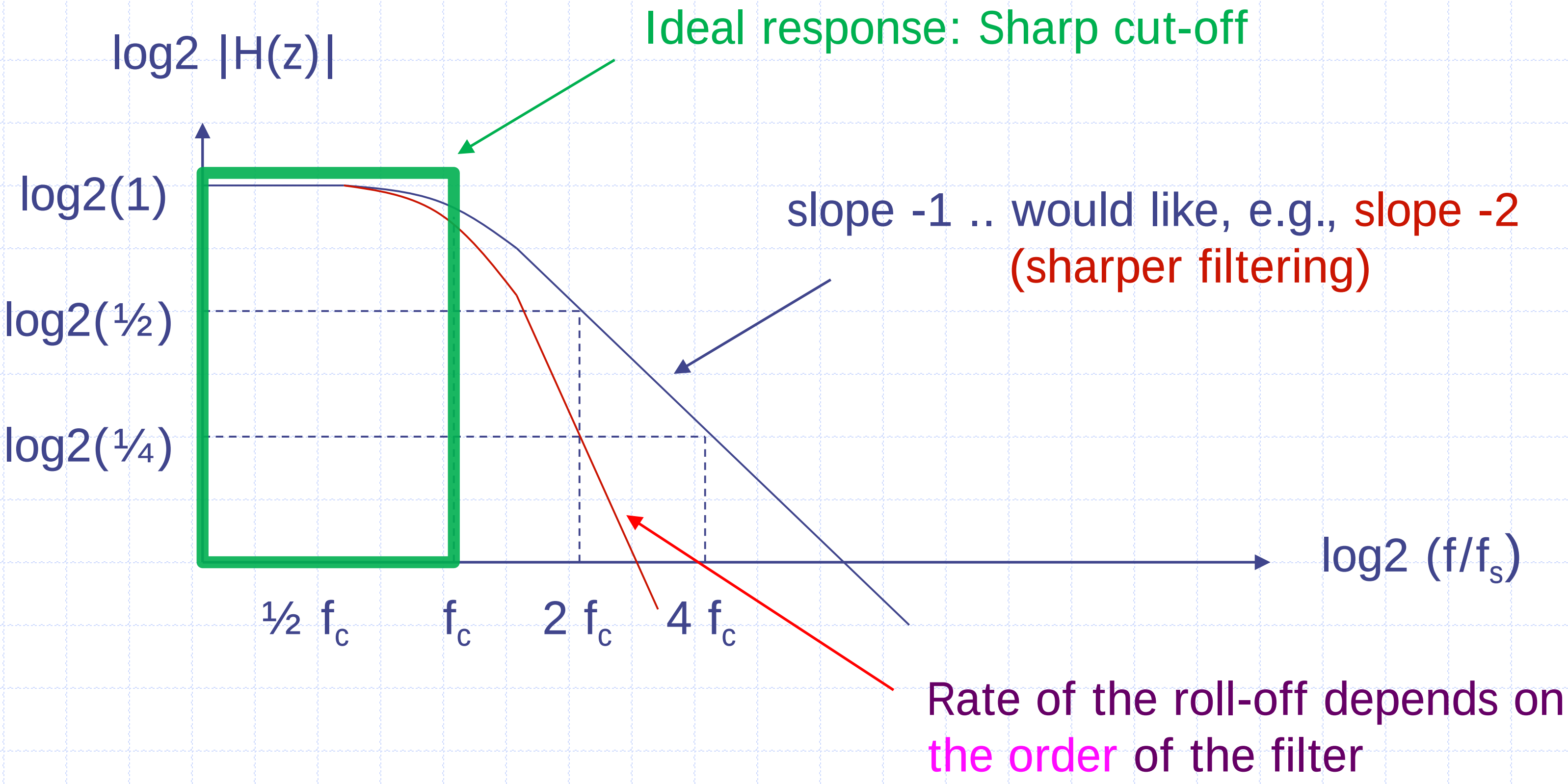
This plot corresponds to the well-known class of **Butterworth** filters (our case: 1^{st}-order Butterworth):

The zero is created by adding x[n-1]:
Previously: y[n] = (N-1)/N y[n-1] + 1/N x[n]
Now: y[n] = (N-1)/N y[n-1] + 1/2N x[n] + 1/2N x[n-1]

y[n] – (N-1)/N y[n-1] = 1/2N x[n] + 1/2N x[n-1]
H(z) = ((z+1)/2N) / (z-(N-1)/N)

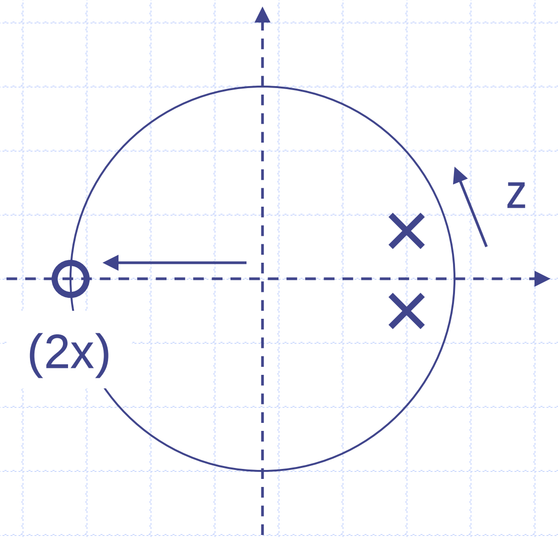
Enhancing Filters

Frequency response 1st-order Butterworth:



Second-order Butterworth

Looking at the pole-zero plot, the IIR filter can be further improved by introducing more poles & zeros.
now $|H(z)|$ has same cut-off freq f_c
but sharper slope!



Computing h[n] (the a_i and b_i)
is difficult, so use a tool to compute coefficients, given f_s and f_c
(Matlab or Web sites)

Just insert found coefficients in IIR equation

$$b_0\, y[n] + b_1\, y[n-1] + b_2\, y[n-2] = a_0\, x[n] + a_1\, x[n-1] + a_2\, x[n-2]$$

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Fixed-point Arithmetic

Why we need it?

- Many microcontrollers have no floating-point unit
- Software floating-point often (too) slow
- Need to implement filters in fixed-point arithmetic

To define a fixed-point type conceptually we need:

- Width of the representation;
- Binary point position in the number.

Trade-off range & resolution

Integer (signed) part	•	Fractional part

Fixed-point Arithmetic (cont.)

2's-complement bit representation (e.g., 32 bits, 14 bits fraction):

2^1 2^0 2^{-1} 2^{-2}

3.75: 000000000000000011 11000000000000

0.02: 000000000000000000 00000101001001

0.02: 000000000000000000 0000010100011110 11

-1.5: 000000000000000001 10000000000000 ^ -1 + 1 =>

111111111111111110 10000000000000

Integer (signed) part • Fractional part

18 bits 14 bits

Exercise: 12, 0.125, -10, -10.5

Fixed-point Arithmetic

- Addition, subtraction as usual (e.g., 15+(-5), 15-5 in 8 bits)
- Multiplication: result must be post-processed:
    - make sure intermediate fits in variable! (e.g., 32 bits)
    - shift right by |fraction| and sign-extend

Example multiplication (32 bits, 14 bits fraction):
Let’s try a simple one instead: (-1.5 * 1 in 4 bits)

3.75: 00000000000000001111000000000000 times:
-1.5: 11111111111111111010000000000000 equals:
10100110000000000000000000000000
(value just fits in 32 bits!)
(now shift right by 14 bits and sign-extend):
11111111111111101001100000000000 which is:
-5.625 111111111111111010 01100000000000

Filter Example

- Second-order Butterworth LP Filter f_c = 10Hz, f_s = 1250Hz
- Coefficients:

$a_0 = 0.0006098548$	$a_1 = 2\ a_0$	$a_2 = a_0$
$b_0 = 1$	$b_1 = -1.9289423$	$b_2 = 0.9313817$

Bit representation (e.g., 32 bits, 14 bits fraction):

a[0]	000000000000000000	0000000001001 111111
a[0]	000000000000000000	00000000001010
a[1]	000000000000000000	00000000010100
a[2]	000000000000000000	00000000001010
b[1]	000000000000000001	11101101110100 ^ -1 + 1
b[2]	000000000000000000	11101110011100
b[2]	000000000000000000	1110111001101111

Implementation (high-cost)

```
int mul(int c, int d) {
  int result = c * d;
  return (result >> 14);
}

void filter() {
  y0 = mul(a0,x0) + mul(a1,x1) + mul(a2,x2) -
       mul(b1,y1) - mul(b2,y2);
  x2 = x1; x1 = x0; y2 = y1; y1 = y0;
}
```

Filter Approximation Example

- Second-order Butterworth LP Filter f_c = 10Hz, f_s = 1250Hz
- Coefficients:

$a_0 = 0.0006098548 * 8/10$ $\quad a_1 = 2\ a_0$ $\quad a_2 = a_0$

$b_0 = 1$ $\quad b_1 = -2$ $\quad b_2 = 1$

See if we can "approximate" the coefficients with binary numbers that contain just 1 bit

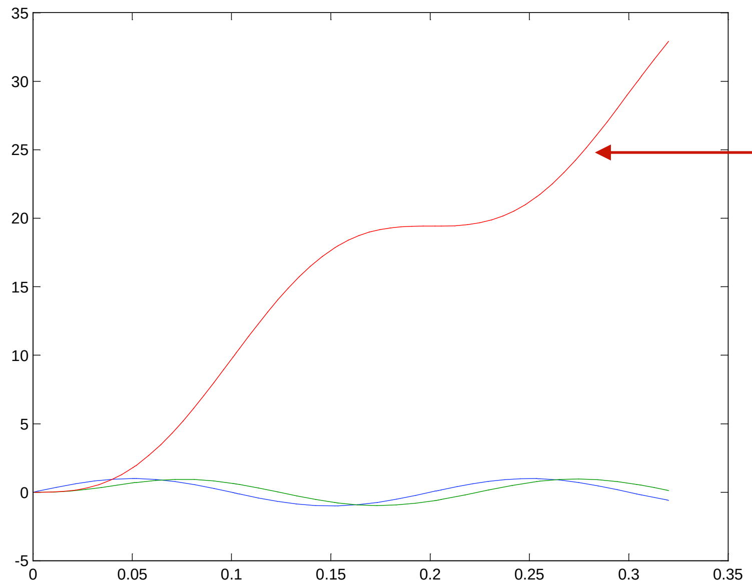
Bit representation (e.g., 32 bits, 14 bits fraction):

a[0]	000000000000000000	00000000001000	(was 10)
a[0]	000000000000000000	00000000001010	Previous value
a[1]	000000000000000000	00000000010000	(was 20)
a[1]	000000000000000000	00000000010100	Previous value
a[2]	000000000000000000	00000000001000	(was 10)
a[2]	000000000000000000	00000000001010	Previous value
-b[1]	000000000000000010	00000000000000	(was 31604)
b[2]	000000000000000001	00000000000000	(was 15260)

Implementation (low-cost)

Why we do this? Shifting bits is faster than multiplication operations

```
y0 = (x0 << 3) >> 14 + (x1 << 4) >> 14 +
     (x2 << 3) >> 14 + (y1 << 15) >> 14 –
     (y2 << 14) >> 14; // assume compiler optimizes ...
x2 = x1; x1 = x0; y2 = y1; y1 = y0;
```



Approx too coarse (2^{nd}-order FIR: a_i, b_i very sensitive!)

Cascade two 1^{st}-order filters

- First-order Butterworth LP Filter f_c = 10Hz, f_s = 1250Hz
- Coefficients:

$a_0 = 0.0245221 \qquad a_1 = a_0$

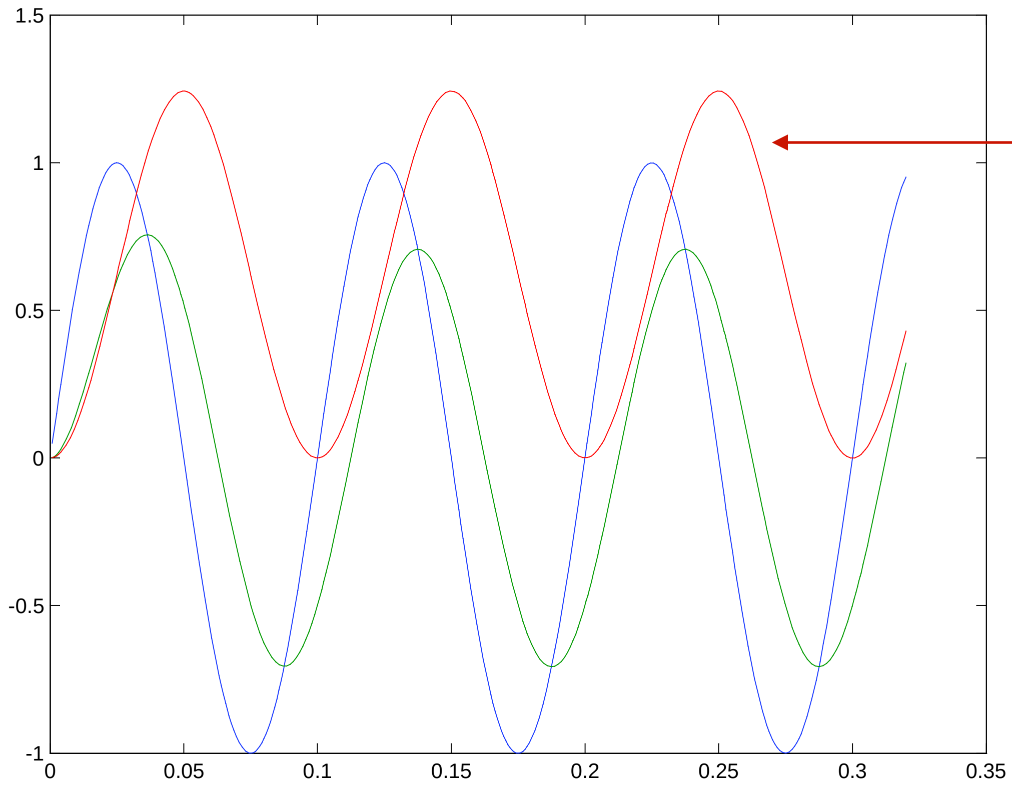
$b_0 = 1 \qquad b_1 = -0.95095676$

Bit representation (e.g., 32 bits, 14 bits fraction):

a[0] 000000000000000000 00000110010010 (a_0 << 14)
a[1] 000000000000000000 00000110010010
b[1] 000000000000000000 11110011011100 ^ -1 + 1

Approx: a[0] = 512 (was 402), b[1] = 16384 (was 15580)

Results



Approx bit better
But still bad for very
low frequencies

So add more powers
of two until good approx
(see matlab demo)

Scaling: tips and tricks

- One size fits all? NO!
  - number of bits depends on needed precision (sensor vs. joystick)

  - special case for proportional controller: P * ε
  - $fp_n * fp_n = fp_{2n}$ (overflow! requires an additional shift)
  - $scalar * fp_n = fp_n$ (overflow? no shift needed)
  - $fp_m * fp_n = fp_{m+n}$ (when P can't be represented as a scalar, using different representations)

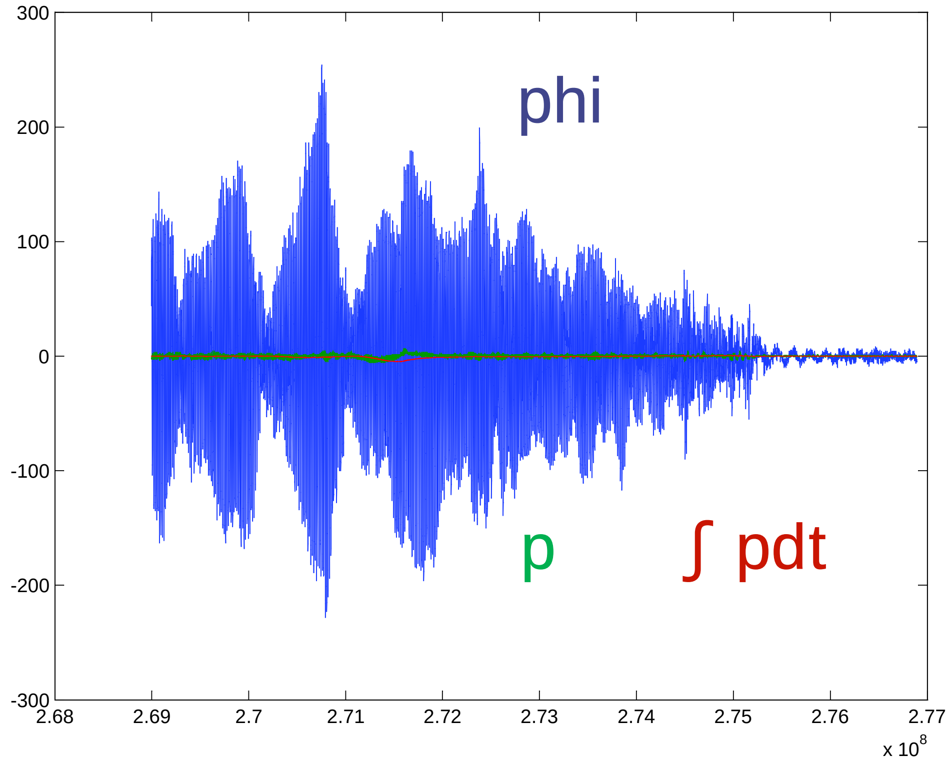
  - document precision for every data type (part of softw arch)

- fp_n to scalar
  - be patient, shift at last instant (when feeding the engines)

Outline

- Introduction
- Z Transform
- FIR Filters
- IIR Filters
- Fixed-point Implementation
- Kalman Filter

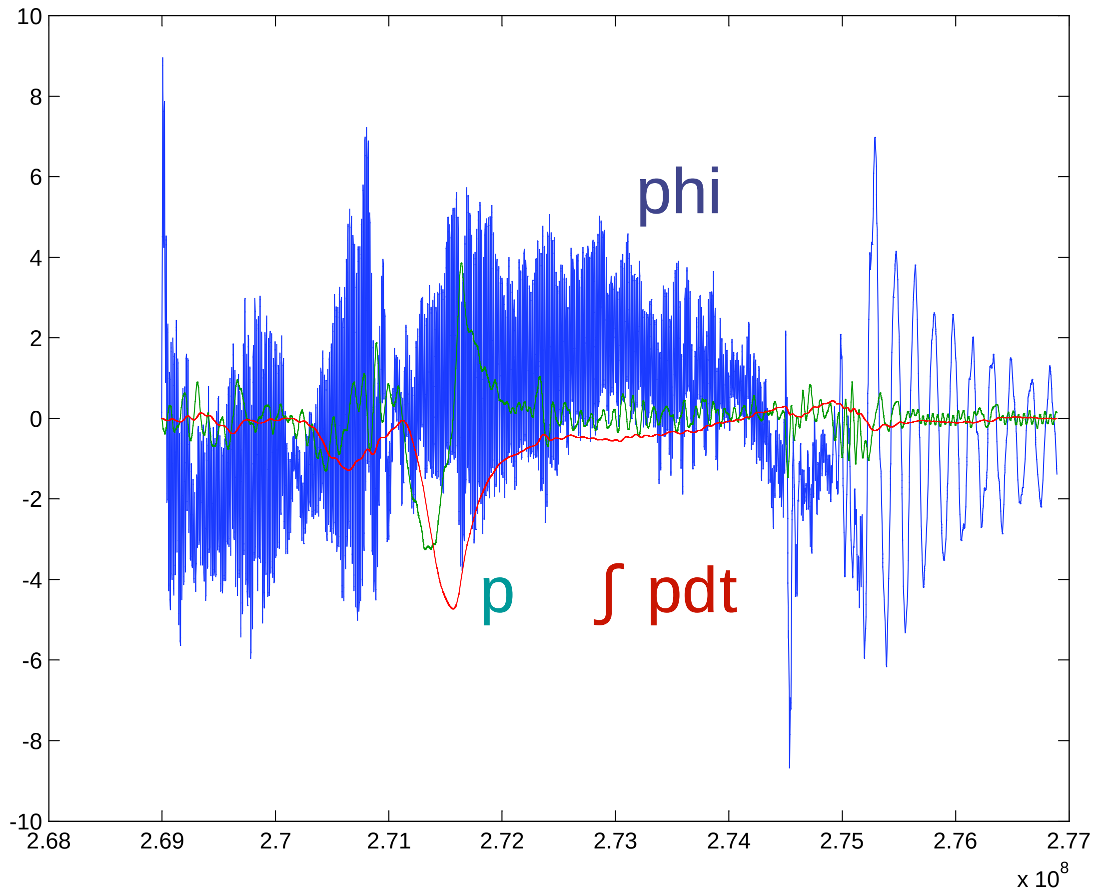
Recall: QR Sensor Signals phi, p



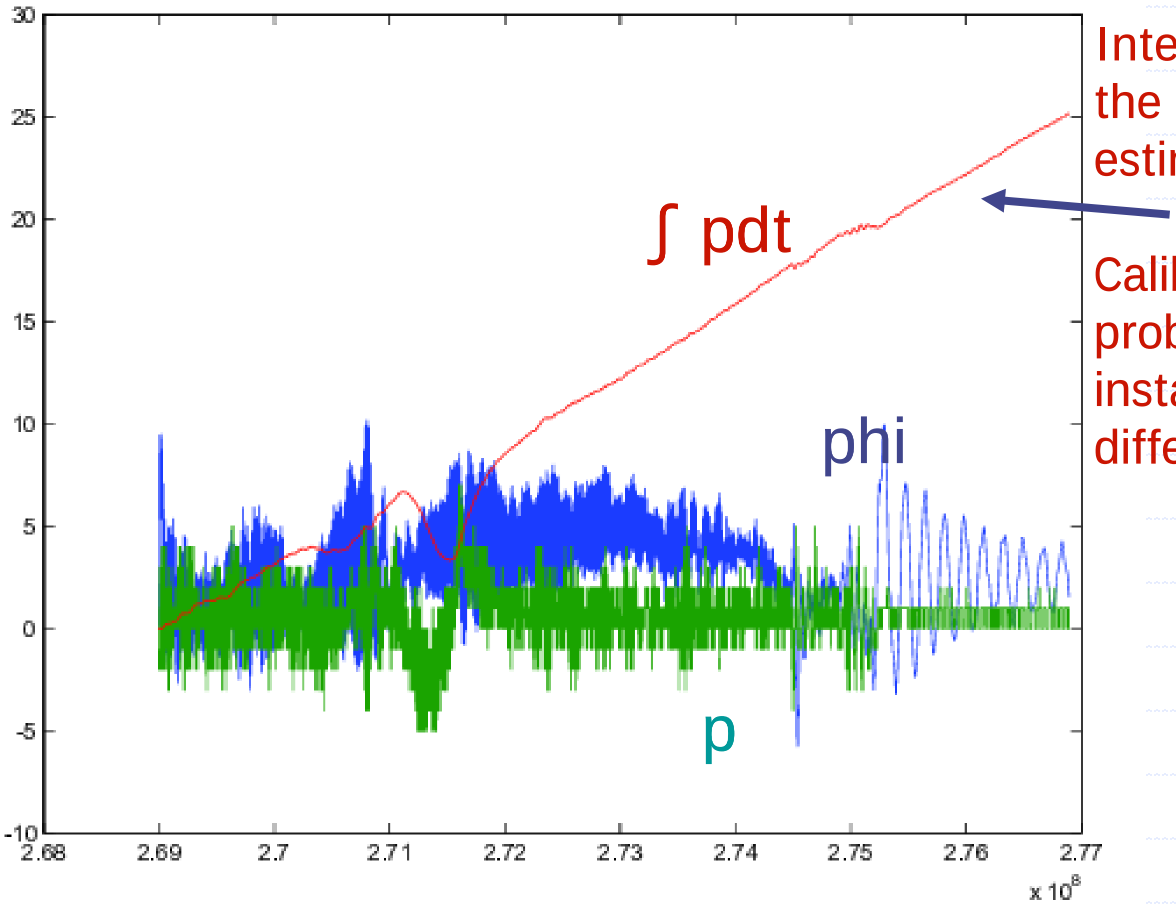
Signal from the previous version of QR:

- phi is the plot of Acc
  - Roll angle
- p is the roll rate
- red is the
  - Integral of roll rate = roll angle

After 2nd-order Low-pass (10Hz)



Bias in p: Integration drift in phi



Integration of the readings from the gyro will lead to drift in the angle estimation over long-term.

Calibrations may not solve the problem all the time, as the bias instability in gyro can lead to different drifts (e.g., temperature)

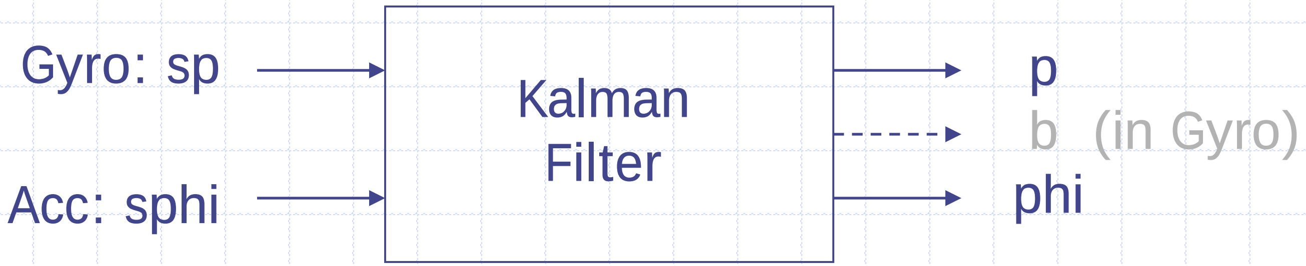
Problem Analysis

- Noise is still considerable
- Still little correlation between (filtered) phi and p
- More aggressive filtering -> more phase delay
- 10 Hz signals already 90 deg phase lag with 2nd-order
- In our particular case we might apply *notch filter*
- In general though, too many noise frequencies
- sphi: negligible drift, too high noise
- sp: low noise, drift -> prohibits integration to phi

- Kalman Filter: combine the best of both worlds!

Kalman Filter (quadcopter near-hover)

- Sensor Fusing: gyro and accel share same information



- Integrate sp to phi
- *Adjust* integration for sp (drift) bias *b* by comparing phi to sphi, averaged over *long* period (phi ~ constant)
  - You have a very large filtering window, and your quadcopter is near-hover, you suppose to get a constant (or Zero) angle (phi ~ constant)!
- Return phi, and p (= sp – bias)

Algorithm

- p = sp – b // estimate real p
- phi = phi + p * P2PHI // integration to predict phi
- e = phi – sphi // compare to measured phi
- phi = phi – e / C1 // correct phi to *some* extent
- b = b + (e/P2PHI) / C2 // adjust bias term

- Initiation b obtain from calibration
- P2PHI: depends on loop freq -> compute/measure
- C1 small: believe sphi ; C1 large: believe sp
- C2 large (typically > 1,000 C1): slow drift

Summary

- DSP is everywhere
- This was merely introduction into the field
- Get a feel for it when applying to QR