In4073
Embedded Real-Time Systems

Introduction to Control Theory
Outline

- Introduction
- Feedback Control
- Blessings of Feedback
- Harnessing Feedback
- QR Control
Why Control Theory?

- Embedded systems integrated with appl’n
- Multi-disciplinary training required:
  - Physics engineering
  - Electronics engineering
  - Mechanical engineering
  - Control engineering
  - ... ... 
  - And, of course,
  - Computer science & engineering
Control is Everywhere

- Automotive
- Aerospace
- Plant Control
- Climate Control
- Health Care
- Copiers, Wafer Scanners
- Model Quad Rotors ...

Diagram:
- Embedded controller
- Controlled system
Cruise Control

e = enable [0/1]
T = throttle [%]
F = thrust [N]
v = velocity [m/s]

\(v_{setp}\) = setpoint
\(v_{meas}\) = measured
\(v_{vehicle}\) = actual

disturbances (slope, wind)
Embedded System

v controller

microcontroller

PC + I/O cards

FPGA Board
Objectives of this Crash Course

- Appreciate the benefits of control
- Understand basic control principles
- Communicate with control engineers

- Get you up to speed to do the QR control
Part I: Feedback Control

- What is Control
- The Feedback Loop
- Proportional Feedback
Velocity Control

Control function:
try to maintain $v_{meas} = v_{setp}$

$$e = v_{setp} - v_{meas}$$

control function:
try to maintain $v_{meas} = v_{setp}$

$$F_{vehicle}$$

$$v_{vehicle}$$

$$v_{meas}$$

$$T$$

$$v_{controller}$$

$$v_{setp}$$

$$v_{meas}$$

$$v_{vehicle}$$

$$v_{meas}$$

$$v_{vehicle}$$

$$v_{meas}$$

$$v_{meas}$$

disturbances (slope, wind)
Feedback Control Loop

controller function $T = h_c(\varepsilon)$: adjust $T$ such that $\varepsilon \to 0$

control theory: how to determine function $h_c$
Standard Loop Format

**Standard form:** control $h_s$ through $h_c$ such that $y = x$

$h_c = h_{controller}$
$h_s = h_{system}$
Proportional Control

Let \( h_c(\varepsilon) = P \varepsilon \)

(Steady-state) Analysis:

Let \( h_s(a) = c a \) (i.e. linear system)
Then \( y = c P (x-y) \Rightarrow y = (c P/(c P+1)) x \)
Effect of Loop Gain

\[ y = \frac{P}{P+1} x \]

Loop gain: the larger, the better \((y \approx x)\)
Example: Velocity Control

Analysis:

\[ v_{\text{meas}} = h_{\text{speedometer}}(v_{\text{vehicle}}) \]

If \( P \gg 1 \) then \( v_{\text{meas}} \approx v_{\text{setp}} \)

Consequently, \( v_{\text{vehicle}} \approx h_{\text{speedometer}}^{-1}(v_{\text{setp}}) \)

Ideally, \( h_{\text{speedometer}}(x) = x \)

Result: \( v_{\text{vehicle}} \approx v_{\text{setp}} \)
Example: Variable Amplifier

Analysis:

If $P \cdot A \gg 1$ (i.e. sufficient loop gain) then $z \approx x$

Hence $y \approx (1/A) \cdot x$ (e.g. $A = 0.001 \Rightarrow 1000 \times \text{amp}$)
Part II: Blessings of Feedback

- High Loop Gain: More Robustness
- High Loop Gain: More Linearity
- High Loop Gain: More Speed
More Robustness

Suppose $h_s$ varies with time

$0.98 \pm 49 \rightarrow 0.02 \rightarrow 0.978 \pm 44.1/45.1 = 0.978$

10% change in $h_s$: only $10\%/50 = 0.2\%$ change in $y$

$1.0 \pm 0.022 \rightarrow 0.978 \pm 44.1/45.1 = 0.978$
Example: Velocity Control

For sufficiently high loop gain: $v_{\text{meas}}$ stable ($\approx v_{\text{setp}}$),
Hence $v_{\text{vehicle}} \approx h_{\text{speedometer}}^{-1}(v_{\text{setp}})$, which is stable
More Linearity

Suppose $h_s$ is non-linear function

Analysis:

Let $h_s(a) = c_a \ a \Rightarrow y = \left(\frac{c_a \ P}{c_a \ P + 1}\right) x$

If $c_a \ P \gg 1$ then $y \approx x \Rightarrow y$ is linear with $x$
Example: Audio Amp

Analysis:

If $c_a P A \gg 1$ then $v \approx v_{in}$
Hence $v_{out} \approx 1/A \, v_{in}$ (so linear gain: $1/A$)
More Speed

Vehicle response (slow):
\[ 10\frac{dv(t)}{dt} + v(t) = T(t) \]

Let \( T(t) = 1 \) \( \Rightarrow \)
\[ v(t) = 1 - e^{-t/10} \]

T and v are typically time-varying signals (function of t). Transfer function (h) is not just a proportional gain function but a first-order transfer function:
Example: Velocity Control

In 2 steps of 100 ms same level (0.74) as 10 s w/o feedback
Performance of vehicle has effectively increased ~50 times!
Part III: Harnessing Feedback

- Instability Problem
- Classical Control Theory
Loop Gain Limitations

Analysis: \( y = \frac{P}{P+1} \times \) 

\[ y = \frac{P}{P+1} \times 0.98, \text{ error } = 0.02 \]

Problem: 
P should be infinite for control error to become zero.
In practice however, loop gain must be limited for stability.
Example 1: Integrator Systems

\[ \frac{dp(t)}{dt} = K(t) \quad \frac{d\phi(t)}{dt} = p(t) \]

P ≥ 1: instability!
Cause: each integration adds 90 deg phase lag
So 2 integrators use up all 180 deg budget!
Example 2: Time Latency

Let $h_s: y(t) = a(t-0.5)$  
(i.e., 0.5s delay)

Phase lag of 180 deg at 1 Hz causes instability
Phase Lag: examples

- Integration (90 deg):
  - speed -> position, flow -> volume

- First-order system (up to 90 deg):
  - lamp, heating, car velocity, ...

- N-th order system (up to N*90 deg):
  - compositions of 1st-order systems, missiles

- Delay systems (unlimited):
  - humans, computers, sample times, cables, air

Need control theory to analyze, e.g., control stability
Describe \( x(t), \ y(t), \ h_c(t), \ h_s(t) \) in terms of their Laplace transforms \( X(s), \ Y(s), \ H_c(s), \ H_s(s) \), respectively.
For linear system \( h \) it holds \( Y(s) = H(s) \cdot X(s) \) (i.e. composition in time domain reduces to multiplication in the Laplace domain). This allows for easy analysis.
Laplace cheat sheet

- $L[a] = a/s$
- $L[a \, t] = a / s^2$
- $L[a \, f + b \, g] = a \, L[f] + b \, L[g]$
- $L[f'] = s \, F(s) - f(0)$
Example: QR Rate Control (1)

\[ H(s) = \frac{1}{s} \]

Let \( x(t) = 1 \Rightarrow X(s) = \frac{1}{s} \)

\[ Y(s) = H(s) X(s) = \frac{1}{s^2} \]

\[ y(t) = t \]

\[ \frac{dy(t)}{dt} = x(t) \]

Laplace transform:

\[ s Y(s) = X(s) \]

\[ H(s) = Y(s)/X(s) = \frac{1}{s} \]

\[ H(s) = \frac{1}{s} \]

Let \( x(t) = 1 \Rightarrow \]

\[ X(s) = \frac{1}{s} \]

\[ Y(s) = H(s) X(s) = \frac{1}{s^2} \Rightarrow \]

\[ y(t) = t \]
Control System Analysis

\[ Y(s) = H(s) \cdot X(s) \]

Stability: Re(\text{roots } H(s)) < 0
\[ \text{Im(roots } H(s)) \text{ small} \]
Example: QR Rate Control (2)

\[ Y(s) = P \frac{H(s)}{1 + PH(s)} X(s) = H'(s) X(s) \]

\[ H(s) = \frac{1}{s} \]

\[ H'(s) = \frac{P}{s} \frac{1}{1 + \frac{P}{s}} = \frac{1}{s/P + 1} \]

First-order system with time constant \(1/P\)
(root: \(s = -P \Rightarrow Re < 0, \text{Im} = 0\) so stable)
Part IV: QR Control

- Instability Problem
- Cascaded P Control
Rate control using P controller

P controller for roll rate:

\[ \frac{dp(t)}{dt} = K(t) \]

\[ p_s - \varepsilon + p = K(\int p \, dt) \]

P < 1: useless control performance
P \geq 1: stable (for not too high P!)
Angle control using P controller

P controller for roll angle:

\[ \phi_s - \frac{dp(t)}{dt} = K(t) \]

\[ \frac{d\phi(t)}{dt} = p(t) \]

\[ P < 1: \text{useless control performance} \]

\[ P \geq 1: \text{instability} \]
Angle control using cascaded P control

Embedded rate controller “neutralizes” one integrator

Cascaded P Controller: stable (for not too high P1 and P2!)

[kalman_control.pdf]
Summary

- Feedback control offers many advantages
- Is ubiquitous (cars, planes, missiles, QRs ..)
- Potential stability problems
- Need control theory
- This was merely introduction into the field
- Get a feel by applying to QR!