Improving the odds
New lower bounds for Van der Waerden numbers

Marijn J.H. Heule

May 14, 2009
1 Introduction

2 Symmetry and extreme certificates

3 Symmetry and generated certificates

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Definitions

Van der Waerden numbers $W(r, k)$ (1927)

The Van der Waerden number $W(r, k)$ is the smallest $n$ such that partitioning $\{1, \ldots, n\}$ into $r$ sets at least one set must contain an arithmetic progression of length $k$. 

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Definitions

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Example of arithmetic progression of length 3: $\{1, 2, 3\}$
Example of arithmetic progression of length 4: $\{2, 5, 8, 11\}$
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Van der Waerden certificates $W(r, k, n)$

A Van der Waerden certificate $W(r, k, n)$ is a partition of $\{1, \ldots, n\}$ into $r$ sets with no set containing an arithmetic progression of length $k$. 
Finding extreme certificates $W(2, 3, n)$

break symmetry: $\{1\}\{\}$
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branch on: $\{1, 2\} \rightarrow \{1, 2\} \{3\}$
Finding extreme certificates $W(2, 3, n)$

break symmetry: $\{1\}\{\}$

branch on: $\{1,2\}\{\} \rightarrow \{1,2\}\{3\}$

branch on: $\{1,2,4\}\{3\} \rightarrow \{1,2,4\}\{3,6,7\} \rightarrow \{1,2,4,5,8\}\{3,6,7,8\}$
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break symmetry: \{1\}\{ \}

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branch on: \{1,2,4\}\{3\} \rightarrow \{1,2,4\}\{3,6,7\} \rightarrow \{1,2,4,5,8\}\{3,6,7,8\}

backtrack to: \{1,2\}\{3,4\} \rightarrow \{1,2,5\}\{3,4\} \rightarrow \{1,2,5\}\{3,4,8\} \rightarrow \{1,2,5,6\}\{3,4,8\} \rightarrow \{1,2,5,6\}\{3,4,7,8\}
An example certificate $W(3, 4, 17)$ and visualizations

$$C_0 = \{1, 3, 11, 13, 15, 16\}$$
$$C_1 = \{2, 4, 5, 8, 17\}$$
$$C_2 = \{6, 7, 9, 10, 12, 14\}$$
An example certificate $W(3, 4, 17)$ and visualizations

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$C_0 : \rightarrow C_1 : \downarrow C_2 : \leftarrow$
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C_0 = \{1, 3, 11, 13, 15, 16\}
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$$
C_0 : \quad C_1 : \quad C_2 :
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An example certificate $W(3, 4, 17)$ and visualizations

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An example certificate $W(3, 4, 17)$ and visualizations

\begin{align*}
C_0 &= \{1, 3, 11, 13, 15, 16\} \\
C_1 &= \{2, 4, 5, 8, 17\} \\
C_2 &= \{6, 7, 9, 10, 12, 14\}
\end{align*}
An example certificate $W(3, 4, 17)$ and visualizations

$C_0 = \{1, 3, 11, 13, 15, 16\}$
$C_1 = \{2, 4, 5, 8, 17\}$
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### Exact numbers and lower bounds

<table>
<thead>
<tr>
<th>$r \setminus k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<tbody>
<tr>
<td>2</td>
<td>9</td>
<td>35</td>
<td>178</td>
<td>1132</td>
<td>&gt;3703</td>
<td>&gt;7484</td>
<td>&gt;27113</td>
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<tr>
<td>3</td>
<td>27</td>
<td>&gt;292</td>
<td>&gt;965</td>
<td>&gt;8886</td>
<td>&gt;43855</td>
<td>&gt;238400</td>
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<tr>
<td>4</td>
<td>76</td>
<td>&gt;1048</td>
<td>&gt;10437</td>
<td>&gt;90306</td>
<td>&gt;387967</td>
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<tr>
<td>5</td>
<td>&gt;125</td>
<td>&gt;2254</td>
<td>&gt;24045</td>
<td>&gt;246956</td>
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<td></td>
<td></td>
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<tr>
<td>6</td>
<td>&gt;207</td>
<td>&gt;9778</td>
<td>&gt;56693</td>
<td></td>
<td></td>
<td></td>
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</tr>
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</table>

V. Chvátal (1970)  
J.R. Rabung (1979)  
R.S. Stevens and R. Shantaram (1978)  
M.D. Beeler and P.E. O’Neil (1979)  
Kouril and Paul (2008)
Extreme certificates $\mathcal{W}(2, 3, 8)$

$C_0 = \{1, 2, 5, 6\}$
$C_1 = \{3, 4, 7, 8\}$

rotation, repetition
Extreme certificates $W(2, 3, 8)$

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$C_0 = \{1, 3, 6, 8\}$
$C_1 = \{2, 4, 5, 7\}$

repetition, reflection
Extreme certificates $W(2, 3, 8)$

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repetition, reflection

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reflection, repetition
Certificates $W(2, 4, 34)$, $W(2, 5, 177)$, and $W(2, 6, 1131)$

$W(2, 4, 34)$ [Chvátal 1970]:

![Diagram of $W(2, 4, 34)$ certificate with purple and white squares]
Certificates $W(2, 4, 34)$, $W(2, 5, 177)$, and $W(2, 6, 1131)$

$W(2, 4, 34)$ [Chvátal 1970]:

![Certificate Image]
Certificates $W(2, 4, 34)$, $W(2, 5, 177)$, and $W(2, 6, 1131)$

$W(2, 4, 34)$ [Chvátal 1970]:

```
  1 0 0 0 1 1 0 1 1 0 1 0 1 1 1 1 0 0 0 0 1 0 1 1 0 0 1 1 1 1 1
  0 1 0 0 1 1 1 1 0 0 0 0 0 0 1 1 0 0 1 0 1 1 1 0 1 0 1 0 0 1
  0 0 1 1 0 0 1 1 1 1 0 0 0 0 1 1 0 1 1 1 1 0 0 1 0 0 0 0 0 1
  0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

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Certificates $W(2, 4, 34)$, $W(2, 5, 177)$, and $W(2, 6, 1131)$

$W(2, 4, 34)$ [Chvátal 1970]:

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Certificates \( W(2, 4, 34), \ W(2, 5, 177), \) and \( W(2, 6, 1131) \)

\( W(2, 4, 34) \) [Chvátal 1970]:

\[ \times 4 + 1 \]

\( W(2, 5, 177) \) [Stevens and Shantaram 1978]:

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$W(2, 6, 1131)$ [Kouril and Franco 2005]:

\[
\begin{array}{c}
\times 4 + 1 \\
\times 4 + 2, \\
\times 5 + 1
\end{array}
\]
Extreme certificates $W(3, 3, 26)$ [Chvátal 1970]
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Observation: For all known Van der Waerden numbers $W(r, k)$ there exists a symmetric certificate $W(r, k, W(r, k) - 1)$.

Extreme certificates $W(4, 3, 75)$ [Beeler and O’Neil]
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Generate certificates $W(r, k, m(k-1)+1)$ [HHvLvM '07]

The general idea:

- Generate a cyclic certificate $W(r, k, m)$
- Repeat it $k - 1$ times and add one last element
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E.g. extreme certificate $W(2, 4, 34):$

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Marijn J.H. Heule  Improving the odds
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[Diagram of a cyclic certificate]
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The general idea:

- Generate a *cyclic* certificate $W(r, k, m)$
- Repeat it $k - 1$ times and add one last element

E.g. extreme certificate $W(2, 4, 34)$:

![Matrix representation of $W(2, 4, 34)$]

Disclaimer:

- No extreme certificate $W(2, 3, 8)$ can be extended
- No extreme certificate $W(3, 3, 26)$ of this kind exists
Generate certificates $W(r, k, p(k - 1) + 1)$ [Rabung ’79]

- Calculate the primitive root of unity $\rho_p$

$$\rho_p^i \pmod{p} \neq \rho_p^j \pmod{p} \quad 1 \leq i < j \leq p-1 \quad (1)$$
Generate certificates \( W(r, k, p(k - 1) + 1) \) [Rabung ‘79]

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  \rho_p^i \pmod{p} \neq \rho_p^j \pmod{p} \quad 1 \leq i < j \leq p-1 \quad (1)
  \]

- Construct sequence \( S_p \) by
  \[
  S_p(i) = \rho_p^i \pmod{p} \quad 1 \leq i \leq p-1 \quad (2)
  \]
Generate certificates $W(r, k, p(k - 1) + 1)$ [Rabung ’79]

- Calculate the primitive root of unity $\rho_p$
  
  $\rho_p^i (\mod p) \neq \rho_p^j (\mod p) \quad 1 \leq i < j \leq p - 1$  

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- Partition the elements $i \in \{1, \ldots, p - 1\}$ into $r$ sets such that
  
  $S_p(i) \in C_{i(\mod r)} \quad p \in C_?$
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  \[
  S_p(i) \in C_{i \pmod{r}} \quad p \in C_r \quad (3)
  \]

- E.g. extreme certificate $W(4, 3, 75)$, so $p = 37$, $\rho_{37} = 2$, $r = 4$
  \[
  S_{37} = 2, 4, 8, 16, 32, 27, 17, 34, 31, 25, 13, 26, 15, 30, 23, 9, 18, 36, 35, 33, 29, 21, 5, 10, 20, 3, 6, 12, 24, 11, 22, 7, 14, 28, 19, 1
  \]
Multiply the elements in \( S_p \) by \( q \):

\[
q \times S_p(i) \in C_{i \mod r} \quad pq \in C?
\]

\[
i \in C_j \Rightarrow i + p \mod pq \in C_{j + \lfloor \frac{r}{2} \rfloor \mod r}
\]
Generate certificates $W(r, k, pq(k−1)+1)$ [HHvLvM ’07]

- Multiply the elements in $S_p$ by $q$:
  
  $$q \times S_p(i) \in C_{i \mod r} \quad \text{and} \quad pq \in C?$$
  
  $$i \in C_j \Rightarrow i + p(\mod pq) \in C_{j + \lfloor \frac{r}{2} \rfloor(\mod r)}$$

- Example $p = 11, q = 2, \rho_{11} = 2, r = 4$

  $S_{11} = 2, 4, 8, 5, 10, 9, 7, 3, 6, 1$

  $C_0 = \{3, 4, 5, 12, 20, 22\}$ \hspace{1cm} $C_1 = \{2, 8, 17, 18, 21\}$

  $C_2 = \{1, 9, 11, 14, 15, 16\}$ \hspace{1cm} $C_3 = \{6, 7, 10, 13, 19\}$
Improved lower bounds $W(4, k)$ [HHvLvM 2007]

$W(4, 5) > 17705$
old bound: 10437

$W(4, 6) > 91331$
old bound: 90306

$W(4, 7) > 393469$
old bound: 387967
Improved lower bounds (2) [HMN 2009?]

$W(3, 5) > 2173$
old bound: 965

$W(3, 6) > 11191$
old bound: 8886

$W(5, 5) > 29621$
old bound: 24045
## Minimal encoding certificates $W(r, k, n)$

<table>
<thead>
<tr>
<th>Variables</th>
<th>Range</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{i,j}$</td>
<td>$i \in {0, \ldots, r-1}$</td>
<td>$j \in C_i$</td>
</tr>
<tr>
<td></td>
<td>$j \in {0, \ldots, n-1}$</td>
<td></td>
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<tr>
<th>Clauses</th>
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<tr>
<td>$(x_{0,j} \lor x_{1,j} \lor \cdots \lor x_{r-1,j})$</td>
<td>$j \in {0, \ldots, n-1}$</td>
<td>$j$ is in at least one set</td>
</tr>
<tr>
<td>$(\neg x_{i,a} \lor \neg x_{i,a+d} \lor \cdots \lor \neg x_{i,a+d(k-1)})$</td>
<td>$i \in {0, \ldots, r-1}$</td>
<td>no row may contain an arithmetic progression of length $k$</td>
</tr>
<tr>
<td></td>
<td>$a \in {1, \ldots, n+1-k}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$d \in {1, \ldots, \left\lfloor \frac{n-a}{k-1} \right\rfloor}$</td>
<td></td>
</tr>
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</table>
[Dransfield, Liu, Marek, Truszczynski 2004]:

“It is likely that $W(5, 3)$ is close to 126 (possibly, it is 126), because 125 was the last integer where we were able to find a counterexample despite significant computational effort.”
## Encoding forced symmetry

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</tr>
<tr>
<td>$(\neg x_{s,j} \lor \neg x_{t,j})$</td>
<td>$0 \leq s &lt; t \leq r-1$</td>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>$d \in {1, \ldots, \lfloor \frac{n-a}{k-1} \rfloor }$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(x_{i,j} \lor \neg x_{i+1(\text{mod } r),j+n/ r(\text{mod } n)})$</td>
<td>$i \in {0, \ldots, r-1}$</td>
<td>forcing a $\frac{360^\circ}{r}$ rotation</td>
</tr>
<tr>
<td>$j \in {0, \ldots, n-1}$</td>
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</table>
Certificate $W(5, 3, 170)$

$W(5, 3) > 170$
Analyzing $W(5, 3, 170)$

The first 85 elements:

$C_0 = \{3, 4, 6, 11, 15, 17, 25, 41, 48, 60, 61, 63, 64, 69, 70, 73, 84\}$

$C_1 = \{1, 2, 5, 16, 20, 21, 23, 28, 32, 34, 42, 58, 65, 77, 78, 80, 81\}$

$C_2 = \{9, 10, 12, 13, 18, 19, 22, 33, 37, 38, 40, 45, 49, 51, 59, 75, 82\}$

$C_3 = \{7, 14, 26, 27, 29, 30, 35, 36, 39, 50, 54, 55, 57, 62, 66, 68, 76\}$

$C_4 = \{8, 24, 31, 43, 44, 46, 47, 52, 53, 56, 67, 71, 72, 74, 79, 83, 85\}$
Analyzing $W(5, 3, 170)$

The first 85 elements:

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$C_4 = \{8, 24, 31, 43, 44, 46, 47, 52, 53, 56, 67, 71, 72, 74, 79, 83, 85\}$

$i \in C_j \Rightarrow i + 17(\text{mod } 85) \in C_{j+1}(\text{mod } 5)$
Analyzing $W(5, 3, 170)$

The first 85 elements:

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$C_3 = \{7, 14, 26, 27, 29, 30, 35, 36, 39, 50, 54, 55, 57, 62, 66, 68, 76\}$
$C_4 = \{8, 24, 31, 43, 44, 46, 47, 52, 53, 56, 67, 71, 72, 74, 79, 83, 85\}$

$i \in C_j \Rightarrow i + 17(\text{mod } 85) \in C_{j+1(\text{mod } 5)}$

$S_{17} = 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$
The first 85 elements:

- \( C_0 = \{3, 4, 6, 11, 15, 17, 25, 41, 48, 60, 61, 63, 64, 69, 70, 73, 84\} \)
- \( C_1 = \{1, 2, 5, 16, 20, 21, 23, 28, 32, 34, 42, 58, 65, 77, 78, 80, 81\} \)
- \( C_2 = \{9, 10, 12, 13, 18, 19, 22, 33, 37, 38, 40, 45, 49, 51, 59, 75, 82\} \)
- \( C_3 = \{7, 14, 26, 27, 29, 30, 35, 36, 39, 50, 54, 55, 57, 62, 66, 68, 76\} \)
- \( C_4 = \{8, 24, 31, 43, 44, 46, 47, 52, 53, 56, 67, 71, 72, 74, 79, 83, 85\} \)

\[ i \in C_j \Rightarrow i + 17 \pmod{85} \in C_{j+1} \pmod{5} \]

\( S_{17} = 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1 \)

\[ 5 \times S_{17}(i) \in C_{\pi(i \mod 4)} \quad 85 \in C_4 \]
In case $q \equiv 0 \pmod{5}$

$$q \times S_p(i) \in C_{\pi(i \mod r-1)} \quad (6)$$

$$pq \in C_r \quad (7)$$

$$i \in C_j \Rightarrow i + p \pmod{pq} \in C_{j + \lfloor \frac{r}{2} \rfloor \pmod{r}} \quad (8)$$

• Repeat $k - 1$ times, but one cannot add one more element
Also improved

$W(5, 5, 98740)$
old bound: 29621

$W(5, 6, 540025)$
old bound: 246956
## Best known exact numbers and lower bounds

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M.J.H. Heule and H. van Maaren (2009?)
Improving the odds
New lower bounds for Van der Waerden numbers

Marijn J.H. Heule

May 14, 2009