Decision Making under Uncertainty

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Outline

Today:

1. Introduction to planning under uncertainty
2. Planning under action uncertainty (MDPs)
3. Planning under sensing uncertainty (POMDPs)

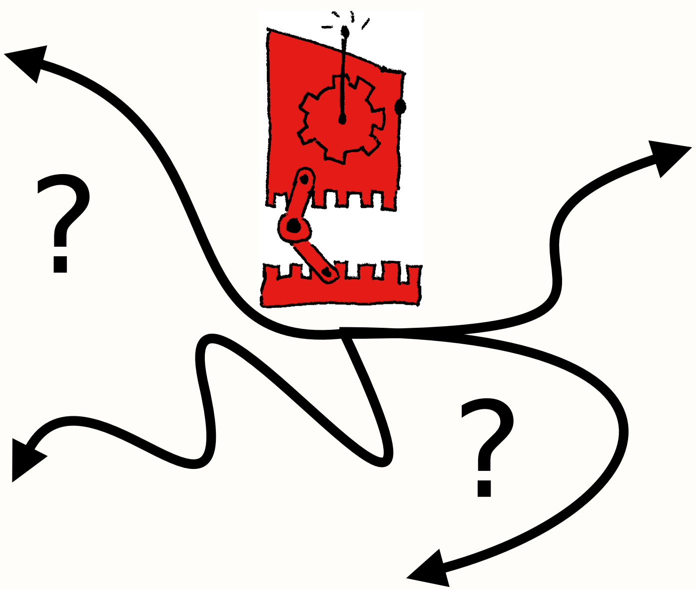
Tomorrow:

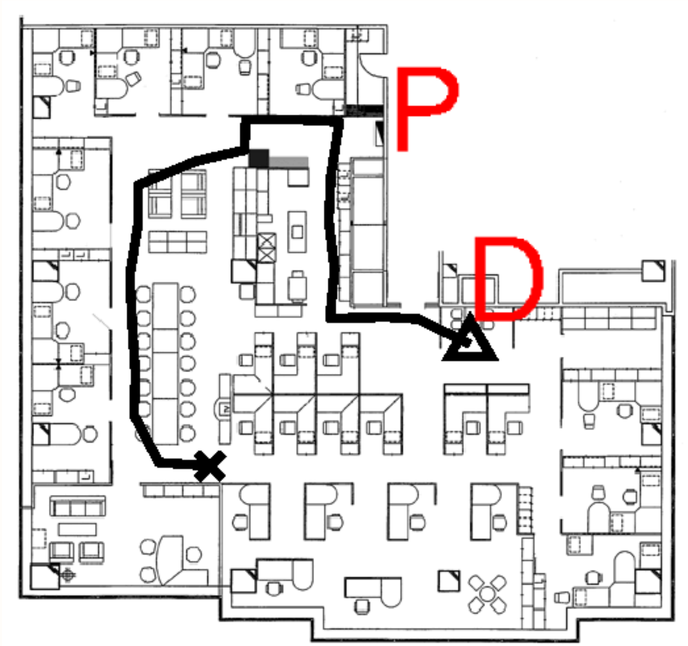
1. Multiagent planning
2. Selected further topics

Introduction

Introduction

- Goal in Artificial Intelligence: to build intelligent agents.
- Our definition of “intelligent”: perform an assigned task as well as possible.
- Problem: how to act?
- We will explicitly model uncertainty.



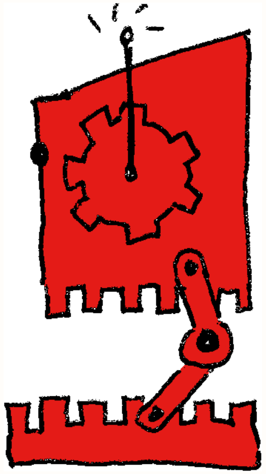


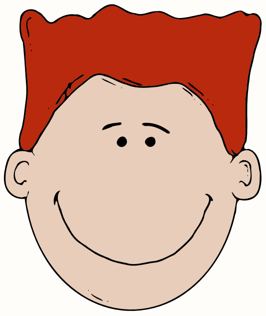
Applications

- Resource planning
- Maintenance
- Queue management
- Medical decision making

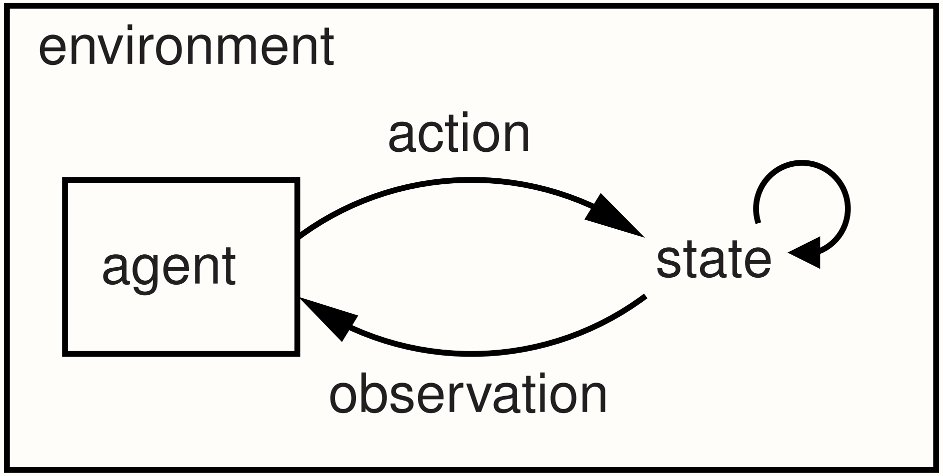
Agents

- An agent is a (rational) decision maker who is able to perceive its external (physical) environment and act autonomously upon it (Russell and Norvig, 2003).
- Rationality means reaching the optimum of a performance measure.
- Examples: humans, robots, some software programs.





Agents



- It is useful to think of agents as being involved in a perception-action loop with their environment.
- But how do we make the right decisions?

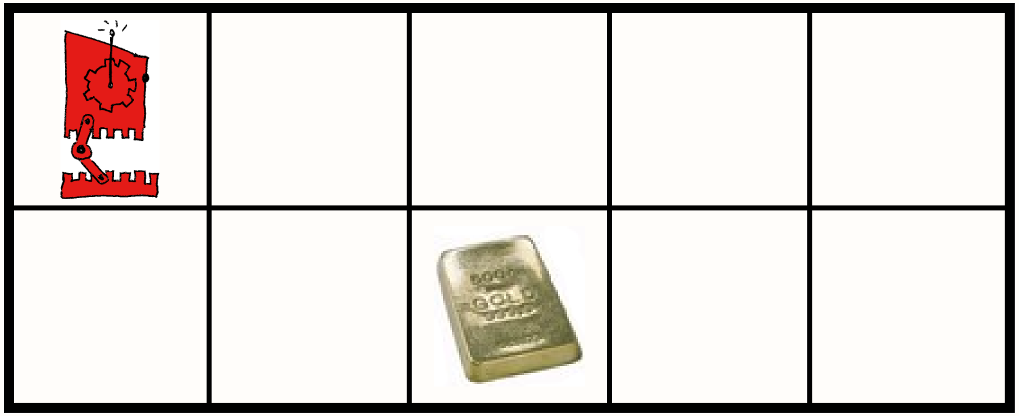
Planning

Planning:

- A plan tells an agent how to act.
- For instance
  - ▶ A sequence of actions to reach a goal.
  - ▶ What to do in a particular situation.
- We need to model:
  - ▶ the agent's actions
  - ▶ its environment
  - ▶ its task

We will model planning as a sequence of decisions.

Classic planning

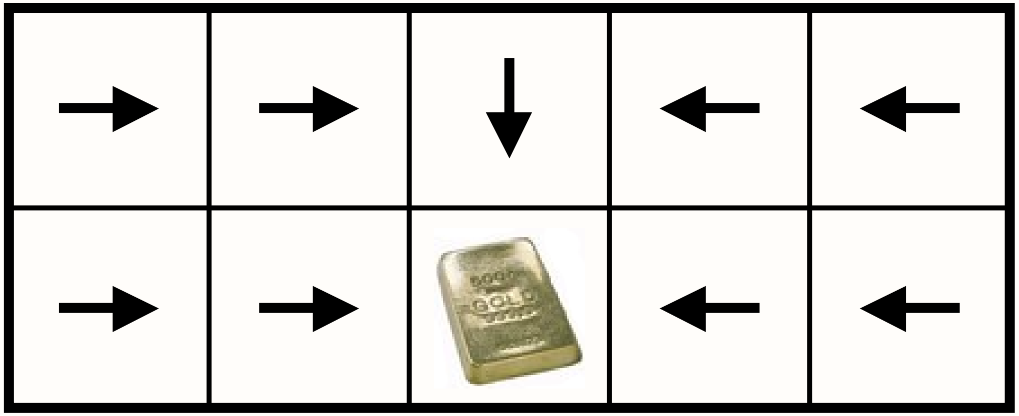


- Classic planning: sequence of actions from start to goal.
- Task: robot should get to gold as quickly as possible.
- Actions: $\rightarrow \downarrow \leftarrow \uparrow$
- Limitations:
  - New plan for each start state.
  - Environment is deterministic.

Classic planning

- Classic planning: sequence of actions from start to goal.
- Task: robot should get to gold as quickly as possible.
- Actions: $\rightarrow \downarrow \leftarrow \uparrow$
- Limitations:
  - New plan for each start state.
  - Environment is deterministic.
- Three optimal plans: $\rightarrow \rightarrow \downarrow$, $\rightarrow \downarrow \rightarrow$, $\downarrow \rightarrow \rightarrow$.

Conditional planning



- Assume our robot has noisy actions (wheel slip, overshoot).
- We need conditional plans.
- Map situations to actions.

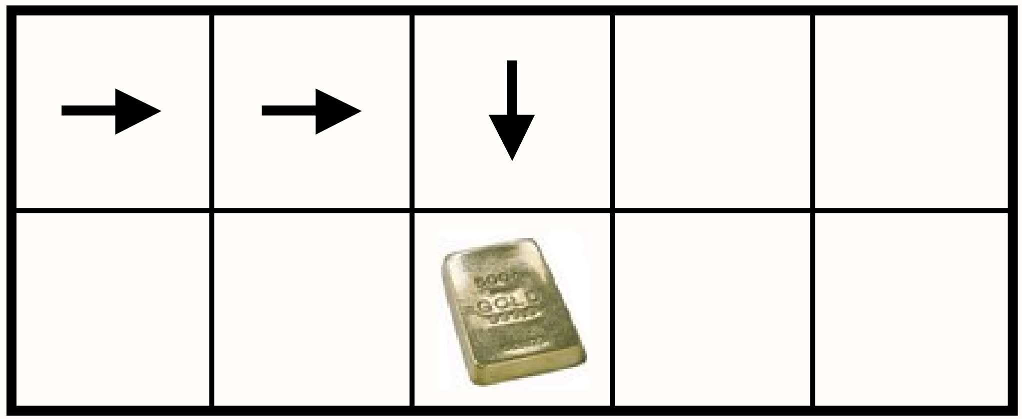
Decision-theoretic planning

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

- Positive reward when reaching goal, small penalty for all other actions.
- Agent's plan maximizes **value**: the sum of future rewards.
- Decision-theoretic planning successfully handles noise in acting and sensing.

Decision-theoretic planning

Plan #1:



Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Decision-theoretic planning

Values of this plan:

?	?	?		
		10		

Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Decision-theoretic planning

Values of this plan:

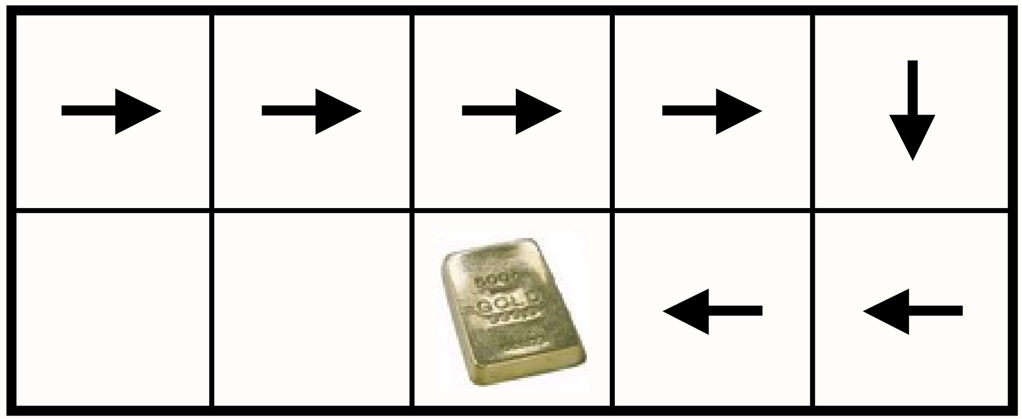
9.7	9.8	9.9		
		10		

Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Decision-theoretic planning

Plan #2:



Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Decision-theoretic planning

Values of this plan:

?	?	?	?	?
		10	?	?

Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Decision-theoretic planning

Values of this plan:

9.3	9.4	9.5	9.6	9.7
		10	9.9	9.8

Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Decision-theoretic planning

Optimal values (encode optimal plan):

9.7	9.8	9.9	9.8	9.7
9.8	9.9	10	9.9	9.8

Reward:

-0.1	-0.1	-0.1	-0.1	-0.1
-0.1	-0.1	10	-0.1	-0.1

Markov Decision Processes

Sequential decision making under uncertainty

- Uncertainty is abundant in **real-world planning** domains.
- **Bayesian** approach $\Rightarrow$ probabilistic models.

Main assumptions:

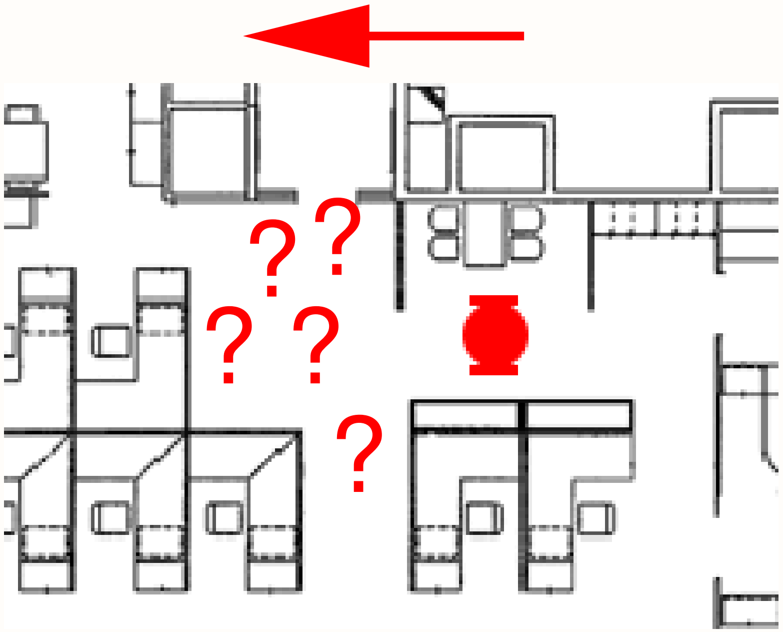
Sequential decisions: problems are formulated as a sequence of "independent" decisions;

Markovian environment: the state at time t depends only on the events at time $t-1$;

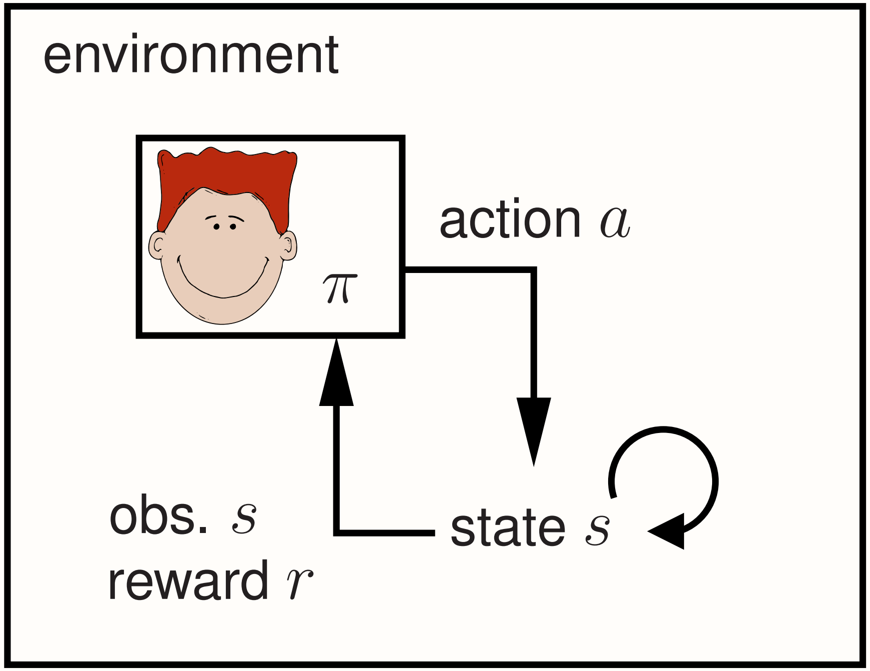
Evaluative feedback: use of a reinforcement signal as performance measure (reinforcement learning);

Transition model

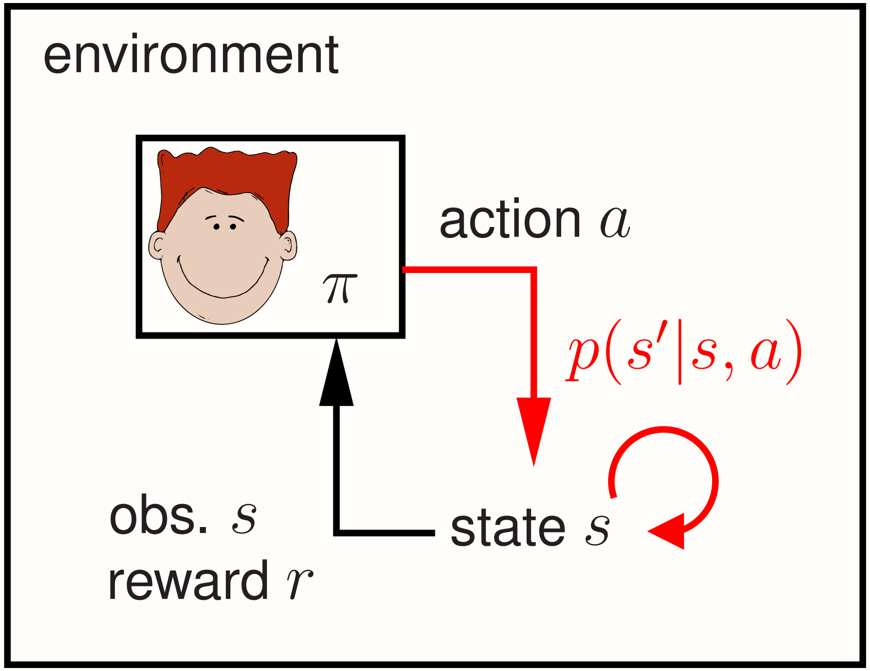
- For instance, robot motion is inaccurate.
- Transitions between states are **stochastic**.
- $p(s'|s,a)$ is the probability to jump from state s to state s' after taking action a.



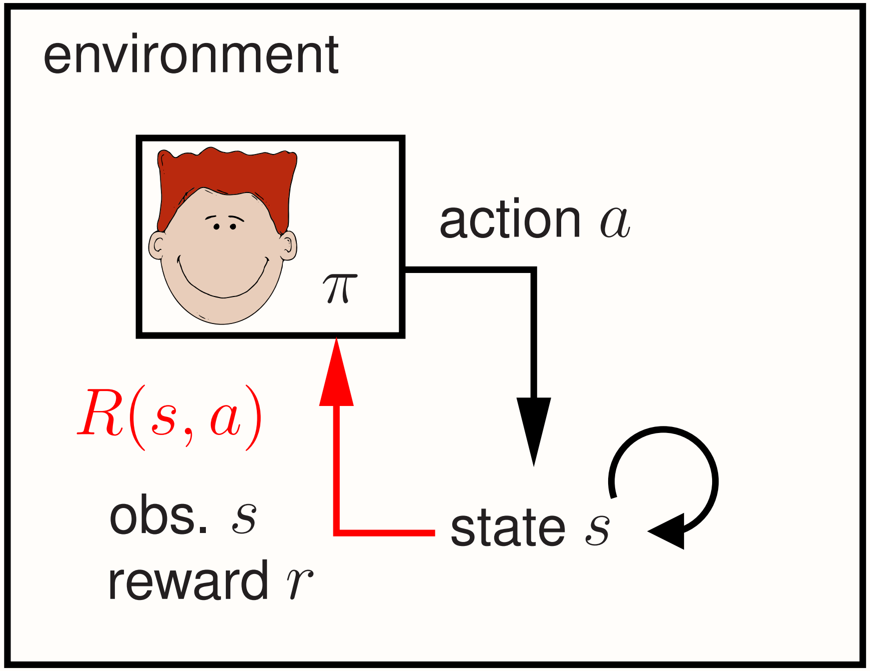
MDP Agent



MDP Agent



MDP Agent



Optimality criterion

For instance, agent should maximize the value

$$E\Big[\sum_{t=0}^{h}\gamma^t R_t\Big], \tag{1}$$

where

- h is the planning horizon, can be finite or ∞
- γ is a discount rate, $0 \leq \gamma < 1$

Reward hypothesis (Sutton and Barto, 1998):
All goals and purposes can be formulated as the maximization of the cumulative sum of a received scalar signal (reward).

Discrete MDP model

Discrete Markov Decision Process model (Puterman, 1994; Bertsekas, 2000):

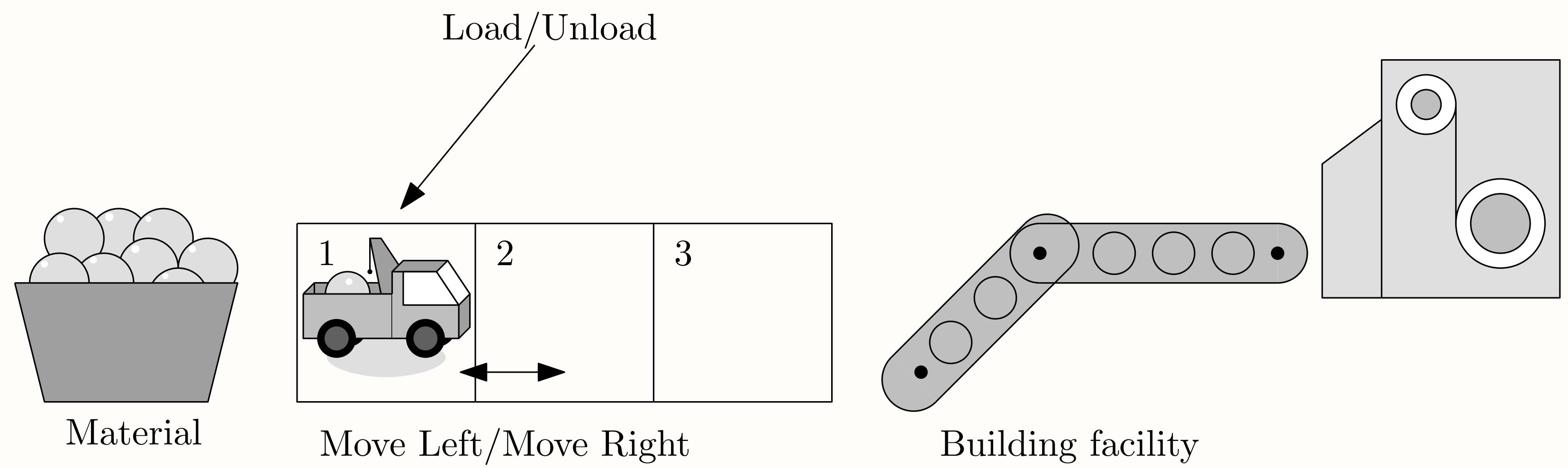
- Time t is discrete.
- State space S.
- Set of actions A.
- Reward function $R : S \times A \mapsto \mathbb{R}$.
- Transition model $p(s'|s,a)$, $T_a : S \times A \mapsto \Delta(S)$.
- Initial state s_0 is drawn from $\Delta(S)$.

The Markov property entails that the next state s_{t+1} only depends on the previous state s_t and action a_t:

$$p(s_{t+1}|s_t, s_{t-1}, \ldots, s_0, a_t, a_{t-1}, \ldots, a_0) = p(s_{t+1}|s_t, a_t). \qquad (2)$$

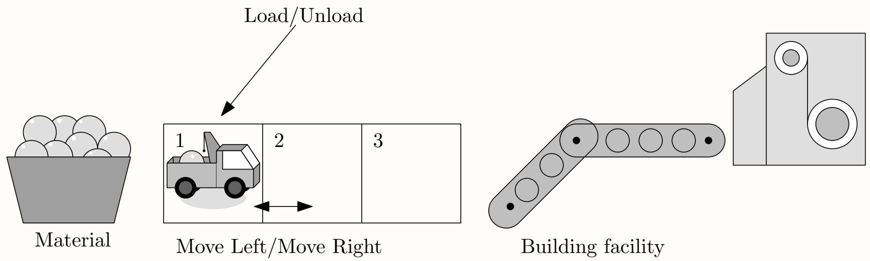
A simple problem

Problem:
An autonomous robot must learn how to transport material from a deposit to a building facility.



(thanks to F. Melo)

Load/Unload as an MDP



- States: $S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$;

  1_U Robot in position 1 (unloaded);

  2_U Robot in position 2 (unloaded);

  3_U Robot in position 3 (unloaded);

  1_L Robot in position 1 (loaded);

  2_L Robot in position 2 (loaded);

  3_L Robot in position 3 (loaded)

- Actions: $A = \{$Left, Right, Load, Unload$\}$;

Load/Unload as an MDP (1)

- Transition probabilities: "Left"/"Right" move the robot in the corresponding direction; "Load" loads material (only in position 1); "Unload" unloads material (only in position 3). Ex:

$$\begin{aligned}(2_L, \text{Right}) &\rightarrow 3_L;\\ (3_L, \text{Unload}) &\rightarrow 3_U;\\ (1_L, \text{Unload}) &\rightarrow 1_L.\end{aligned}$$

- Reward: We assign a reward of $+10$ for every unloaded package (payment for the service).

Load/Unload as an MDP (2)

- For each action $a \in A$, T_a is a matrix.
  Ex:

$$T_{\text{Right}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Recall: $S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$.

Load/Unload as an MDP (3)

- The reward $R(s, a)$ can also be represented as a matrix
  Ex:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +10 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{\text{Left, Right, Load, Unload}\}$

Policies and value

- Policy π: tells the agent how to act.
- A deterministic policy $\pi : S \mapsto A$ is a mapping from states to actions.
- Value: how much reward $E[\sum_{t=0}^{h} \gamma^t R_t]$ does the agent expect to gather.
- Value denoted as $Q^\pi(s, a)$: start in s, do a and follow π afterwards.

Policies and value (1)

- Extracting a policy π from a value function Q is easy:

$$\pi(s) = \arg\max_{a \in A} Q(s, a). \tag{3}$$

- Optimal policy π^*: one that maximizes $E[\sum_{t=0}^{h} \gamma^t R_t]$ (for every state).
- In an infinite-horizon MDP there is always an optimal deterministic stationary (time-independent) policy π^*.
- There can be many optimal policies π^*, but they all share the same optimal value function Q^*.

Dynamic programming

Since S and A are finite, $Q^*(s,a)$ is a matrix.
Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q_1 = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

$$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\},\ A = \{\text{Left, Right, Load, Unload}\}$$

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$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{$Left, Right, Load, Unload$\}$

Dynamic programming

Since S and A are finite, $Q^*(s,a)$ is a matrix.
Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{$Left, Right, Load, Unload$\}$

Dynamic programming

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$$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\},\ A = \{\text{Left, Right, Load, Unload}\}$$

Dynamic programming

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$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{\text{Left, Right, Load, Unload}\}$

Dynamic programming

Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_5 = \begin{bmatrix} 0 & 0 & 8.57 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8.57 & 9.03 & 8.57 & 8.57 \\ 8.57 & 9.5 & 9.03 & 9.03 \\ 9.03 & 9.5 & 9.5 & 10 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{$Left, Right, Load, Unload$\}$

Dynamic programming

Iterations of DP:

$$Q_{20} = \begin{bmatrix} 18.53 & 17.61 & 19.51 & 18.54 \\ 18.53 & 16.73 & 17.61 & 17.61 \\ 17.61 & 16.73 & 16.73 & 16.73 \\ 19.51 & 20.54 & 19.51 & 19.51 \\ 19.51 & 21.62 & 20.54 & 20.54 \\ 20.54 & 21.62 & 21.62 & 26.73 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{\text{Left, Right, Load, Unload}\}$

Dynamic programming

Final Q^* and policy:

$$Q^* = \begin{bmatrix} 30.75 & 29.21 & 32.37 & 30.75 \\ 30.75 & 27.75 & 29.21 & 29.21 \\ 29.21 & 27.75 & 27.75 & 27.75 \\ 32.37 & 34.07 & 32.37 & 32.37 \\ 32.37 & 35.86 & 34.07 & 34.07 \\ 34.07 & 35.86 & 35.86 & 37.75 \end{bmatrix} \qquad \pi^* = \begin{bmatrix} \text{Load} \\ \text{Left} \\ \text{Left} \\ \text{Right} \\ \text{Right} \\ \text{Unload} \end{bmatrix}$$

Value iteration

- Value iteration: successive approximation technique.
- Start with all values set to 0.
- In order to consider one step deeper into the future, i.e., to compute V^*_{n+1} from V^*_n:

$$Q^*_{n+1}(s,a) := R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) \max_{a' \in A} Q^*_n(s',a'), \quad (4)$$

  which is known as the dynamic programming update or Bellman backup.
- Bellman (1957) equation:

$$Q^*(s,a) = R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) \max_{a' \in A} Q^*(s',a'). \quad (5)$$

Value iteration (1)

Initialize Q arbitrarily, e.g., $Q(s,a) = 0, \forall s \in S, a \in A$
repeat
  $\delta \leftarrow 0$
  for all $s \in S, a \in A$ **do**
    $v \leftarrow Q(s,a)$
    $Q(s,a) \leftarrow R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) \max_{a' \in A} Q(s',a')$
    $\delta \leftarrow \max(\delta, |v - Q(s,a)|)$
  end for
until $\delta < \epsilon$
Return Q

Value iteration (2)

Value iteration discussion:

- As $n \to \infty$, value iteration converges.
- Value iteration has converged when the largest update δ in an iteration is below a certain threshold ϵ.
- Exhaustive sweeps are not required for convergence, provided that in the limit all states are visited infinitely often.
- This can be exploited by backing up the most promising states first, known as prioritized sweeping.

Solution methods: MDPs

Model based

- Basic: dynamic programming (Bellman, 1957), value iteration, policy iteration.
- Advanced: prioritized sweeping, function approximators.

Model free, reinforcement learning (Sutton and Barto, 1998)

- Basic: Q-learning, TD(λ), SARSA, actor-critic.
- Advanced: generalization in infinite state spaces, exploration/exploitation issues.

POMDPs

Beyond MDPs

- Real agents cannot directly observe the state.
- Sensors provide partial and noisy information about the world.

Beyond MDPs

- MDPs have been very successful, but requires to have an observable Markovian state.
- Many domains this is impossible (or expensive) to obtain:
- Diagnosis (medical, maintenance)

Beyond MDPs

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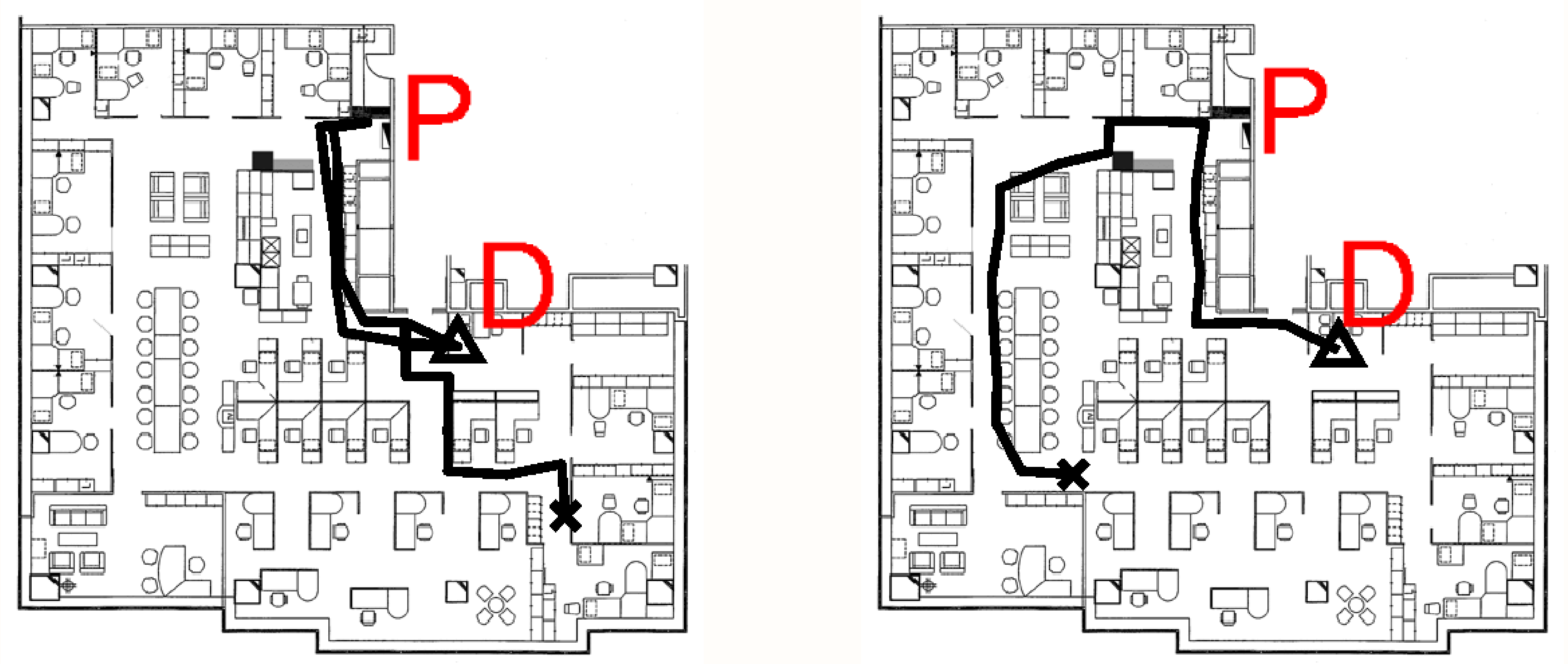
A partially observable problem



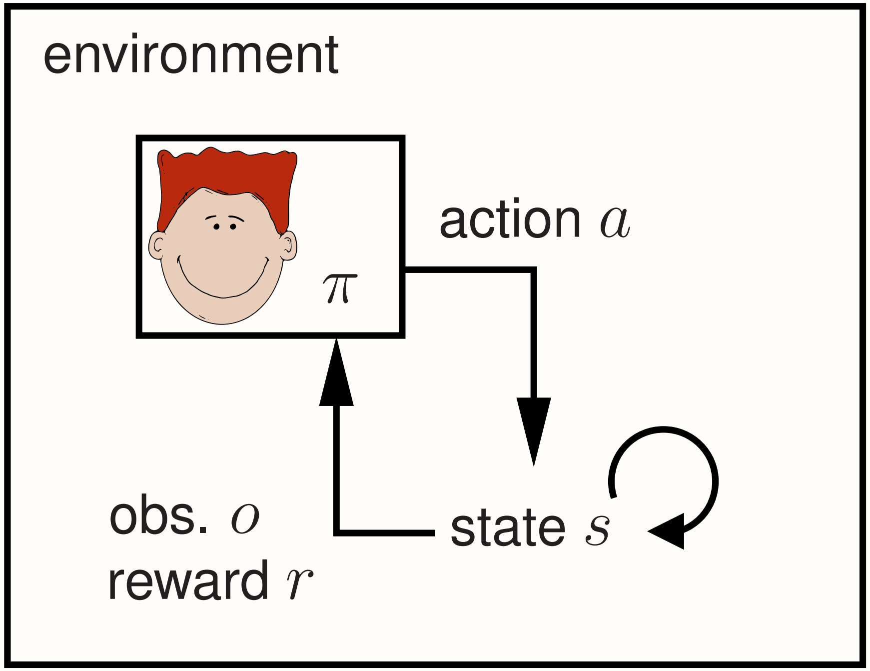
Task: start at random position ($\times$) $\rightarrow$ pick up mail at P $\rightarrow$ deliver mail at D ($\triangle$).

Characteristics: motion noise, perceptual aliasing.

Observation model

- Imperfect sensors.
- Partially observable environment:
  - Sensors are **noisy**.
  - Sensors have a **limited view**.
- $p(o|s', a)$ is the probability the agent receives observation o in state s' after taking action a.

POMDP Agent



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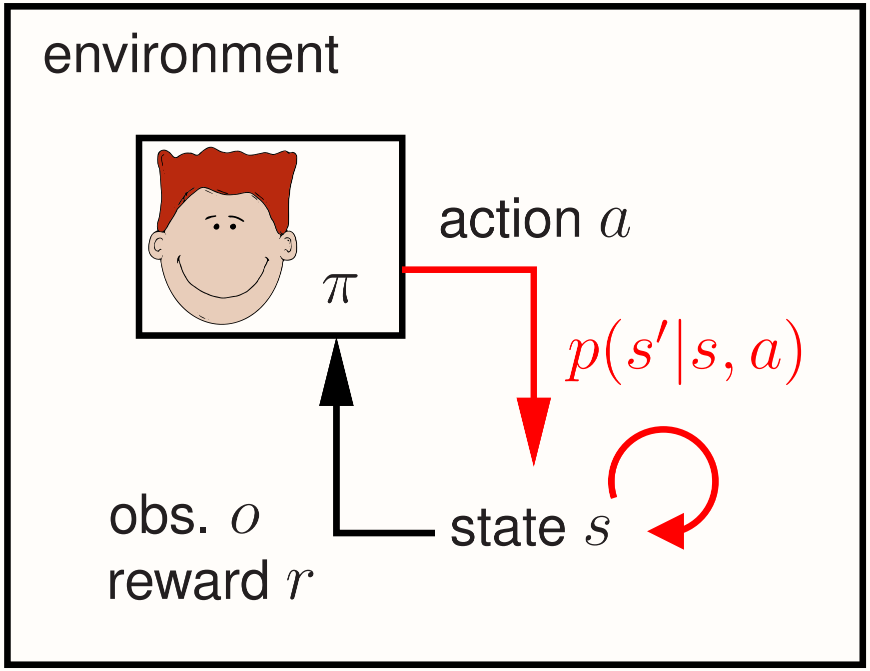

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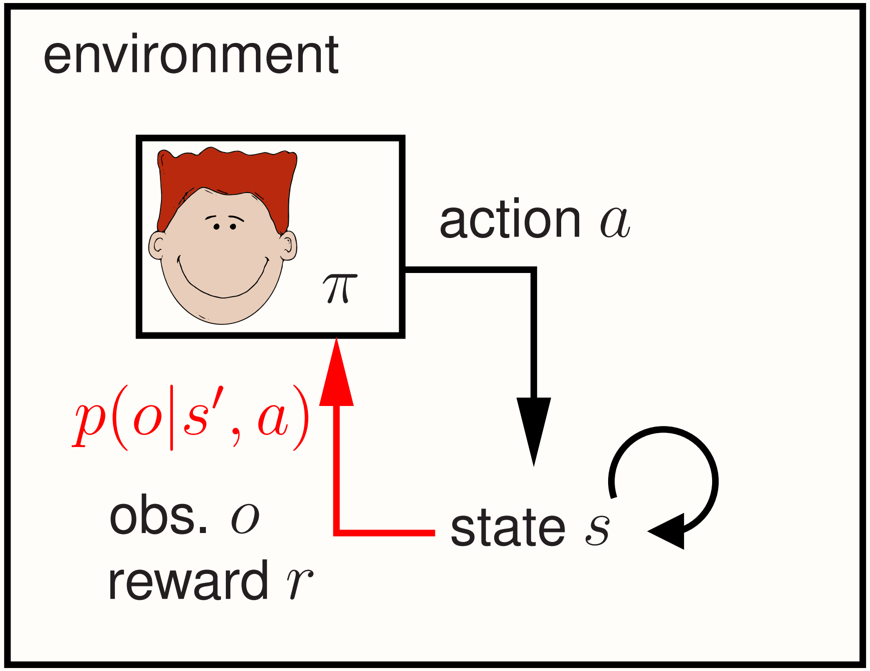

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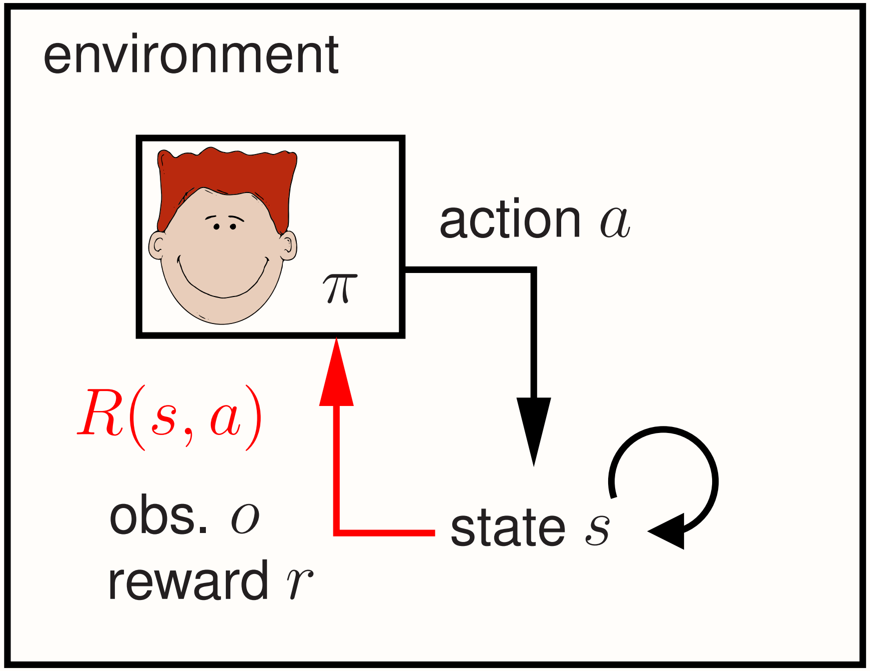

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Partially observable Markov decision processes (POMDPs) (Kaelbling et al., 1998):

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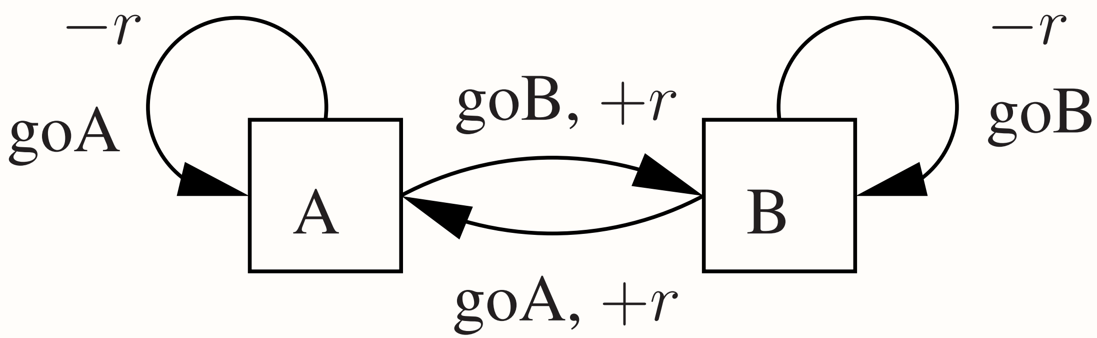
- Goal is to compute plan, or **policy** π, that maximizes long-term reward.

Memory

- In POMDPs memory is required for optimal decision making.
- In this non-observable example (Singh et al., 1994):



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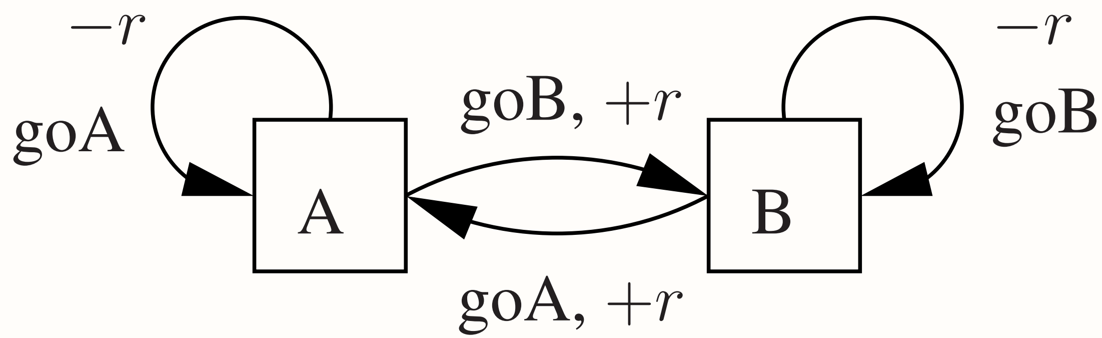

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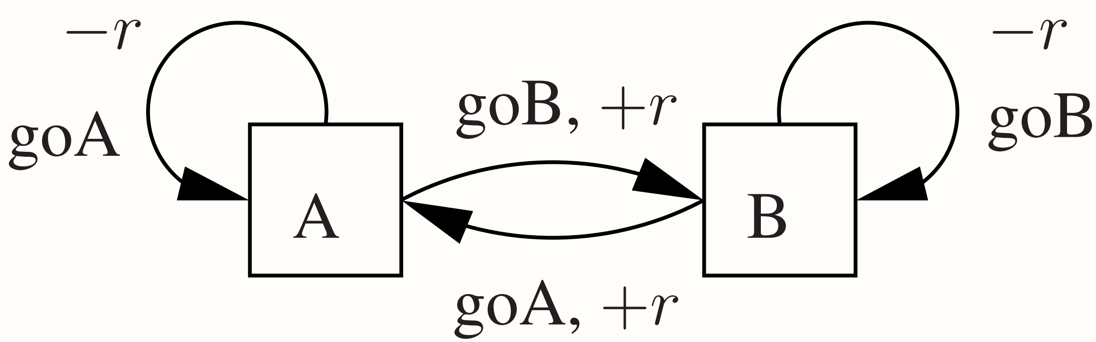

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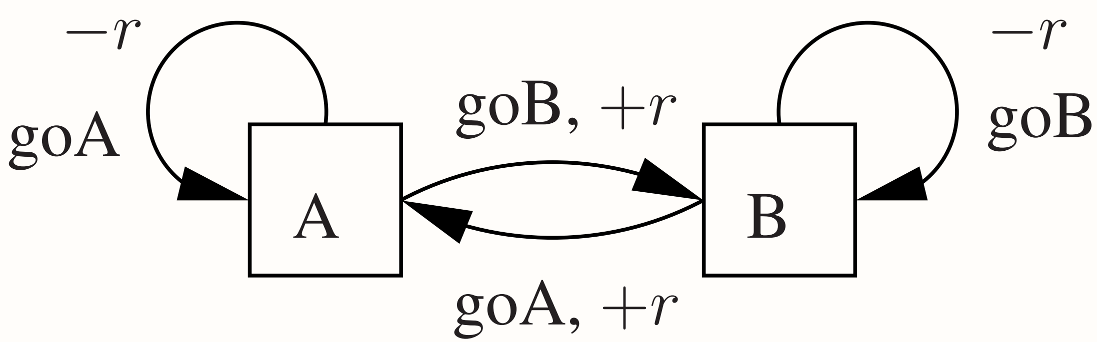

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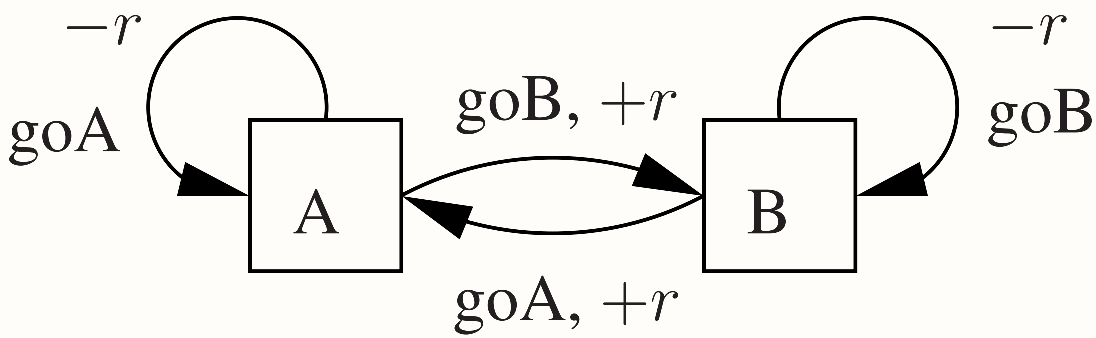

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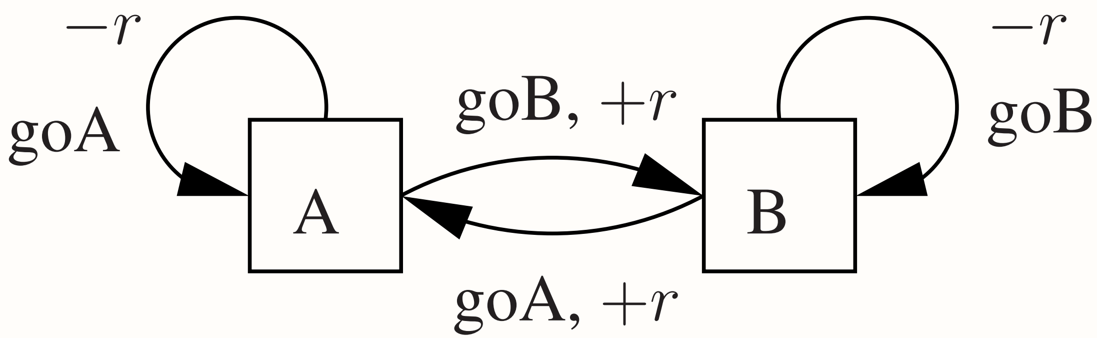

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POMDP: memory-based (optimal)	$V_{\text{min}} = \frac{\gamma r}{1-\gamma} - r$

Beliefs

Beliefs:

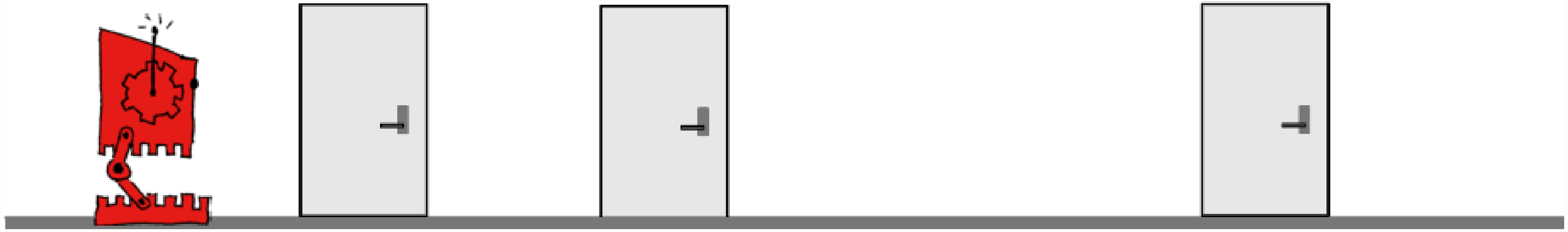
- The agent maintains a **belief** $b(s)$ of being at state s.
- After action $a \in A$ and observation $o \in O$ the belief $b(s)$ can be updated using Bayes' rule:

$$b'(s') \propto p(o|s') \sum_{s} p(s'|s,a) b(s)$$

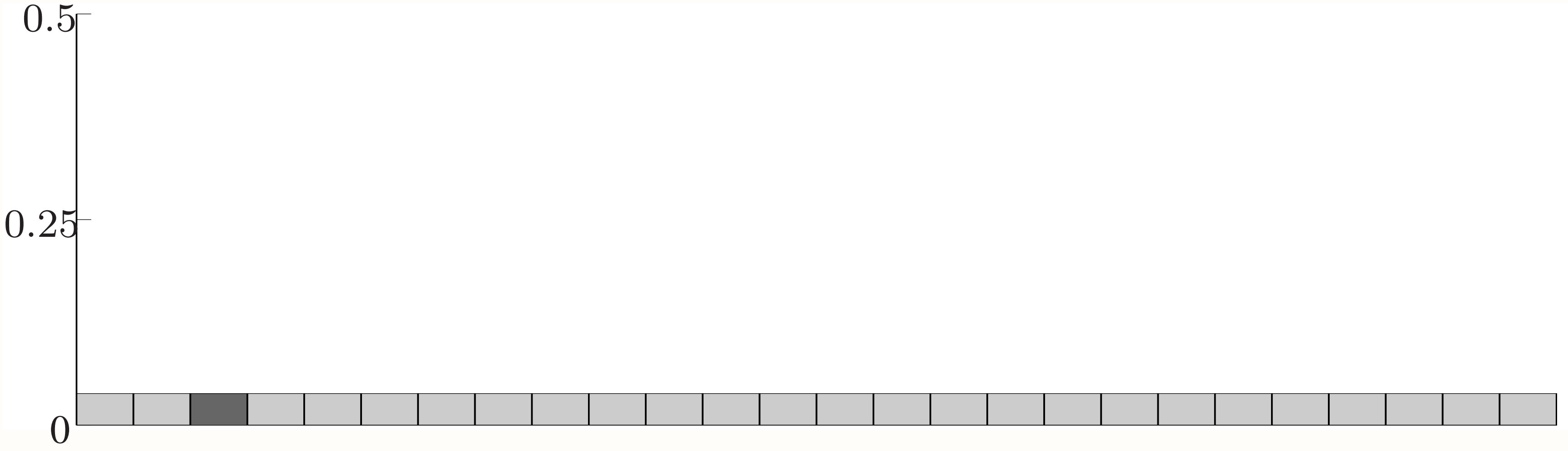
- The belief vector is a **Markov** signal for the planning task.

Belief update example

True situation:



Robot's belief:



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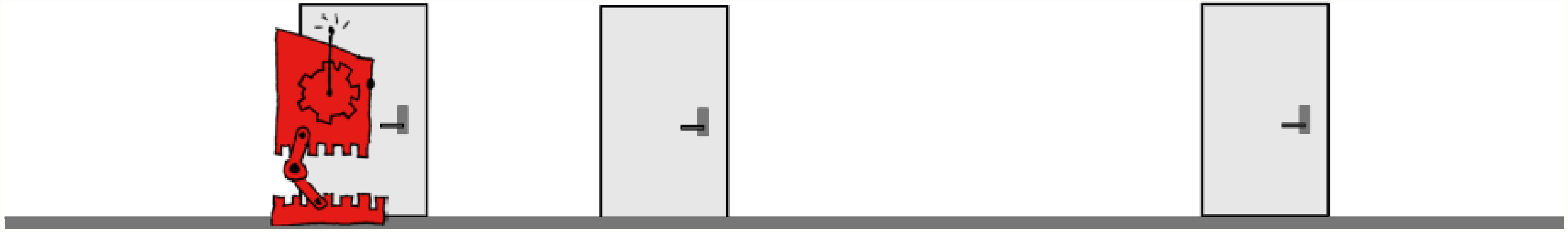

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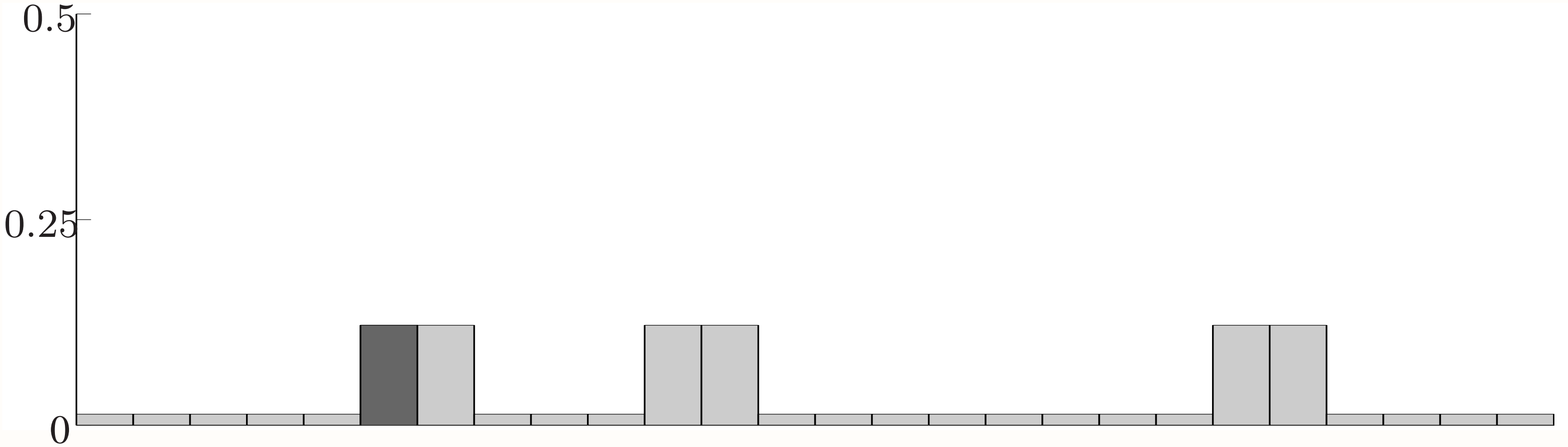

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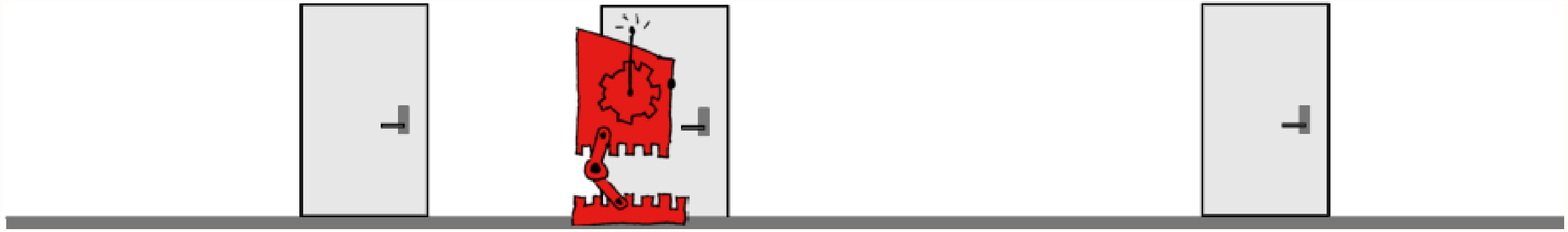

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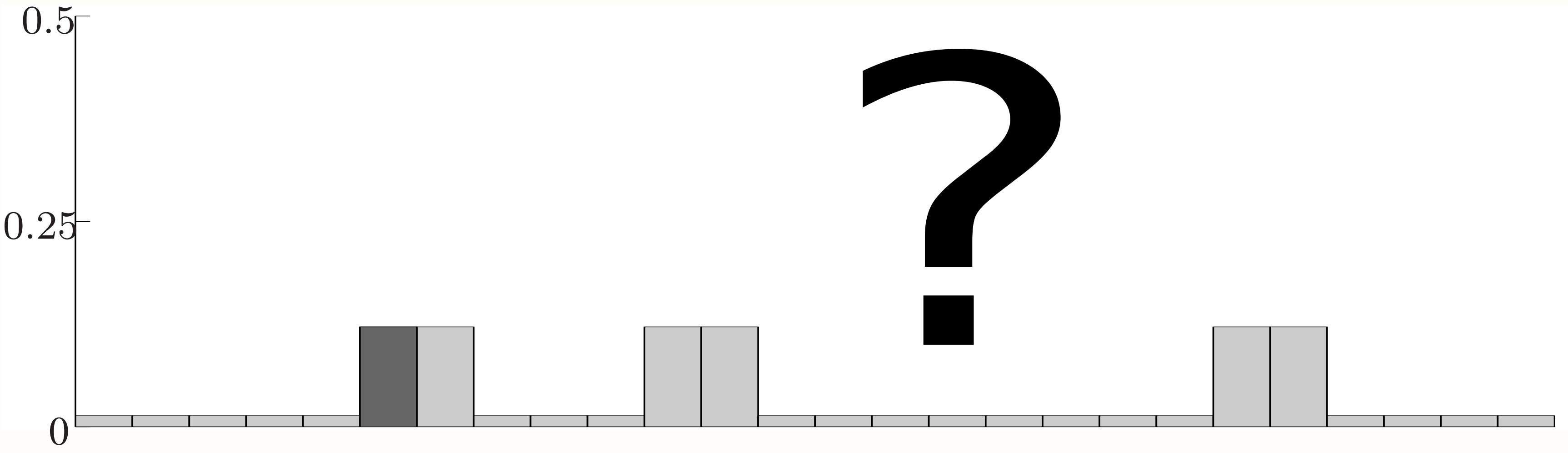

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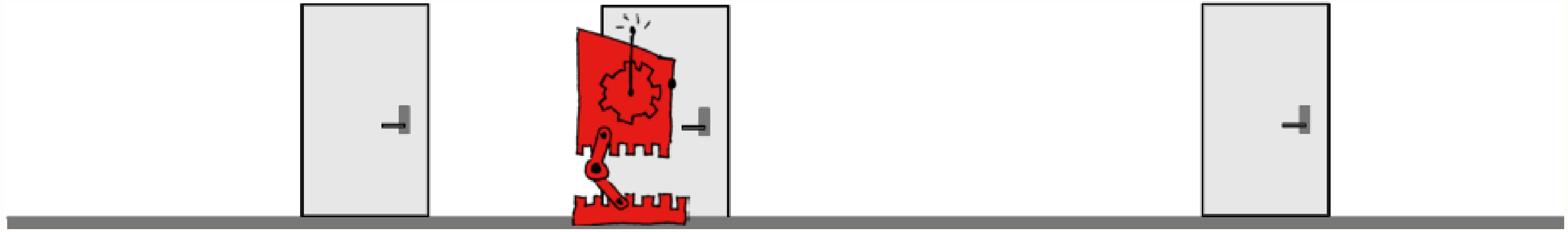

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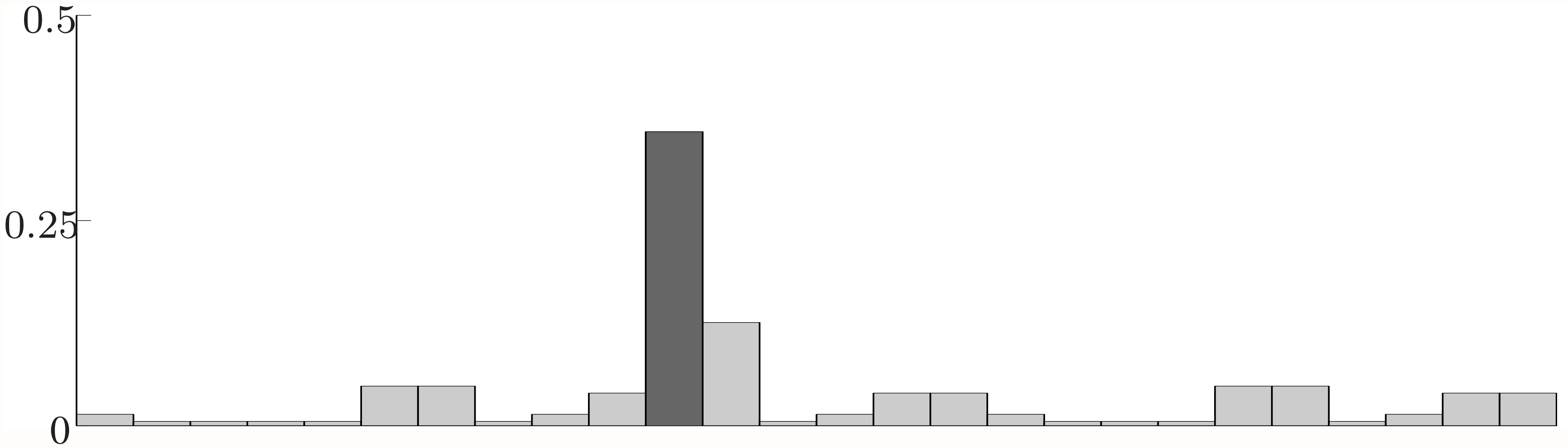

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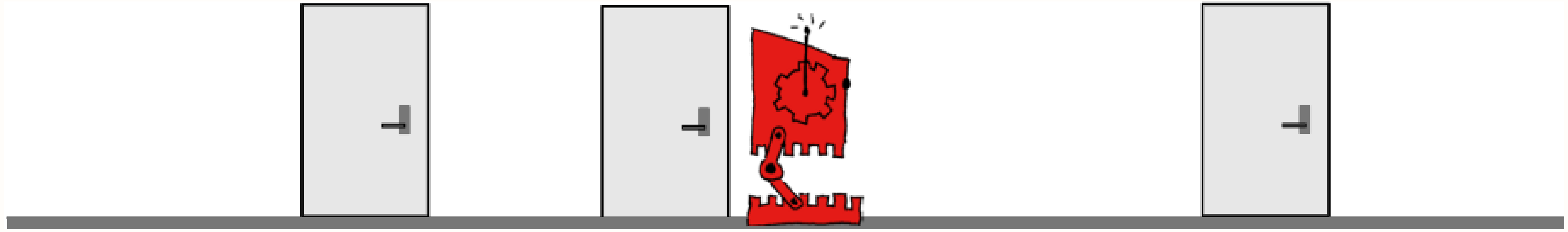

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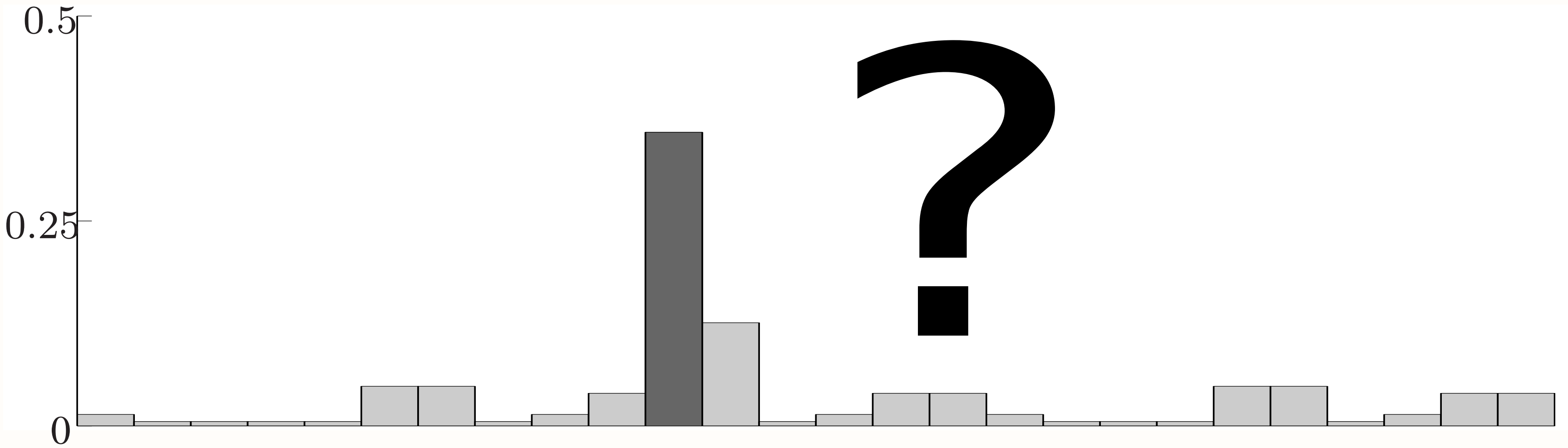

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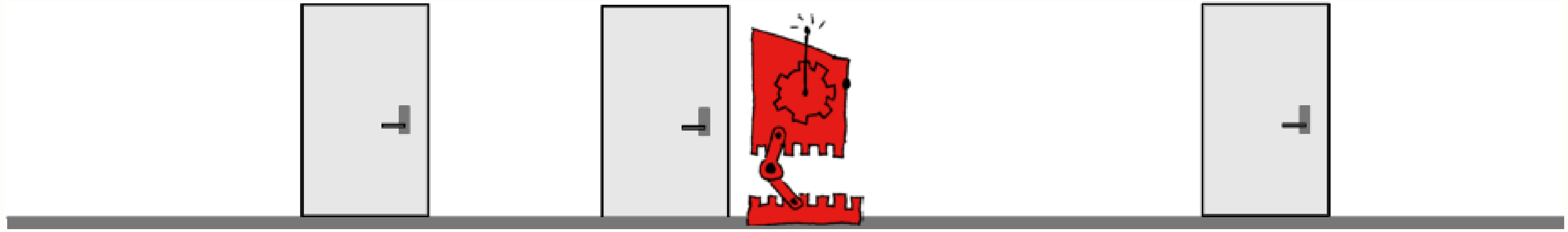

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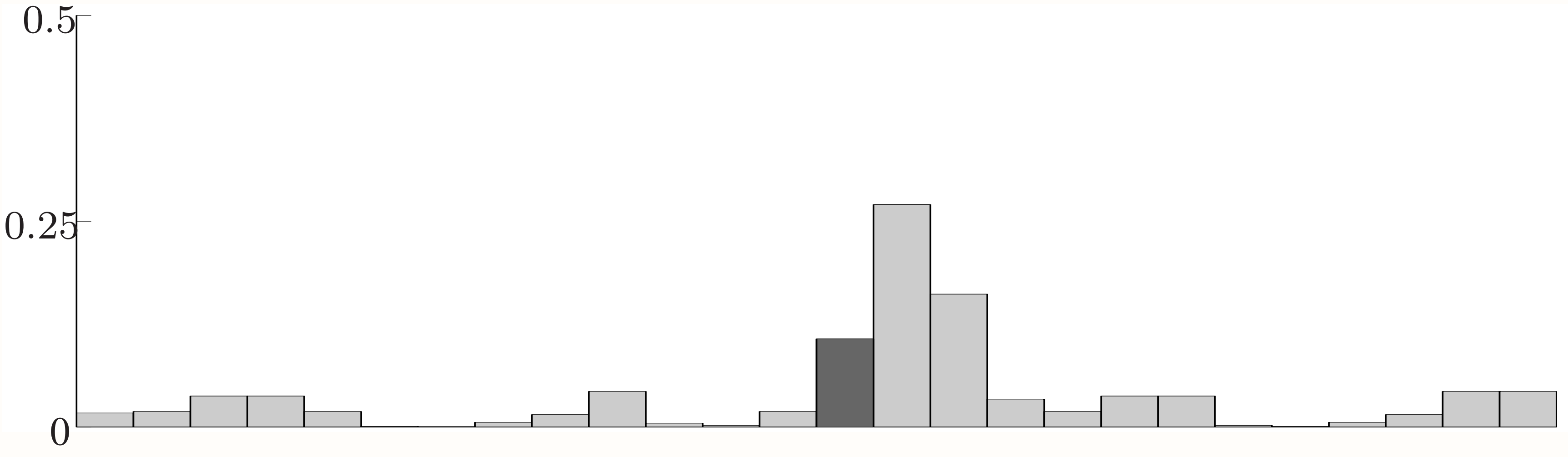

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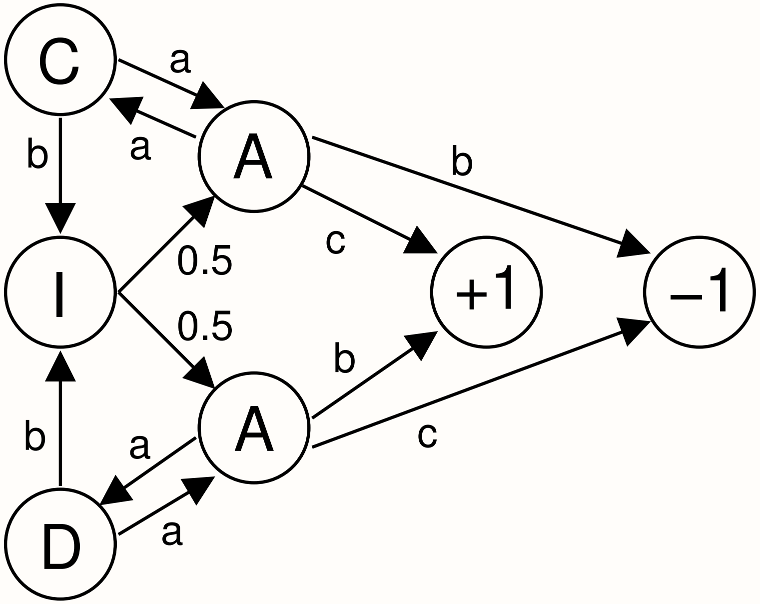
MDP-based algorithms

- Exploit belief state, and use the MDP solution as a heuristic.
- Most likely state (Cassandra et al., 1996):
  $\pi_{MLS}(b) = \pi^*(\arg\max_s b(s))$.
- Q_{MDP} (Littman et al., 1995):
  $\pi_{Q_{\text{MDP}}}(b) = \arg\max_a \sum_s b(s) Q^*(s, a)$.



(Parr and Russell, 1995)

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- The robot fully 'observes' the new belief-state b_a^o after executing a and observing o.

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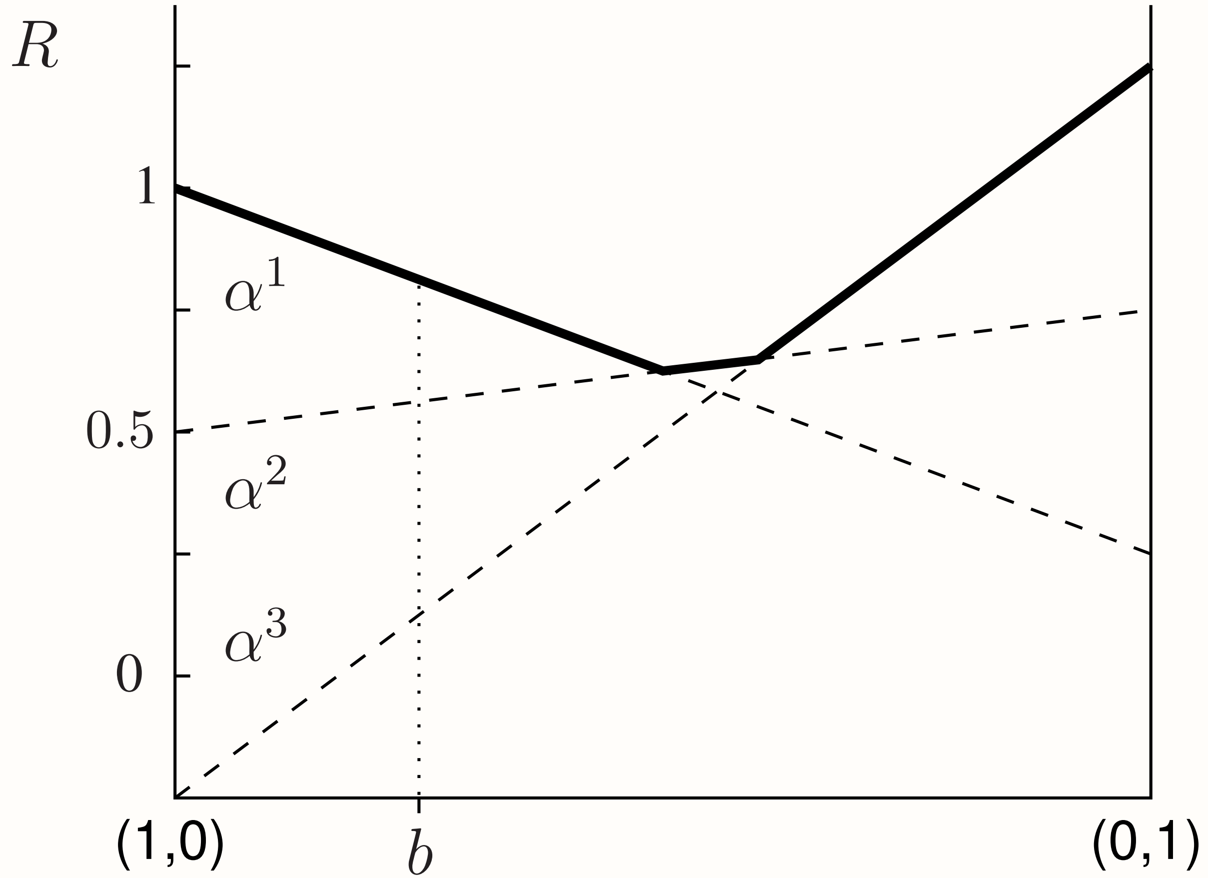
Example V_0



$R(s,a)$	a_1	a_2	a_3
s_1	1.00	0.50	−0.25
s_2	0.25	0.75	1.25

PWLC shape of V_n

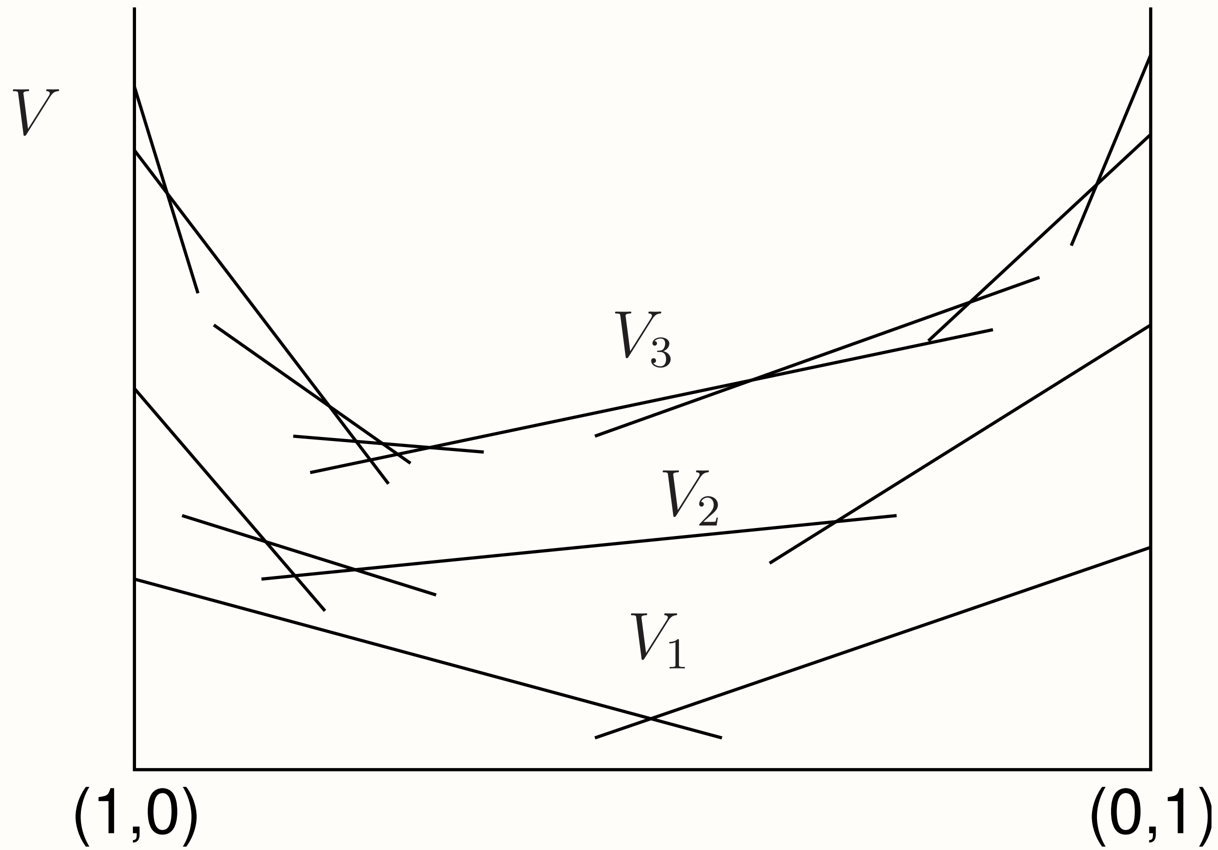
- Like V_0, V_n is as well piecewise linear and convex.
- Rewards $R(b, a) = b \cdot R(s, a)$ are linear functions of b. Note that the value of a point b satisfies:

$$V_{n+1}(b) = \max_a \left[b \cdot R(s, a) + \gamma \sum_o p(o|b, a) V_n(b_a^o) \right]$$

  which involves a maximization over (at least) the vectors $R(s, a)$.
- Intuitively: less uncertainty about the state (low-entropy beliefs) means better decisions (thus higher value).

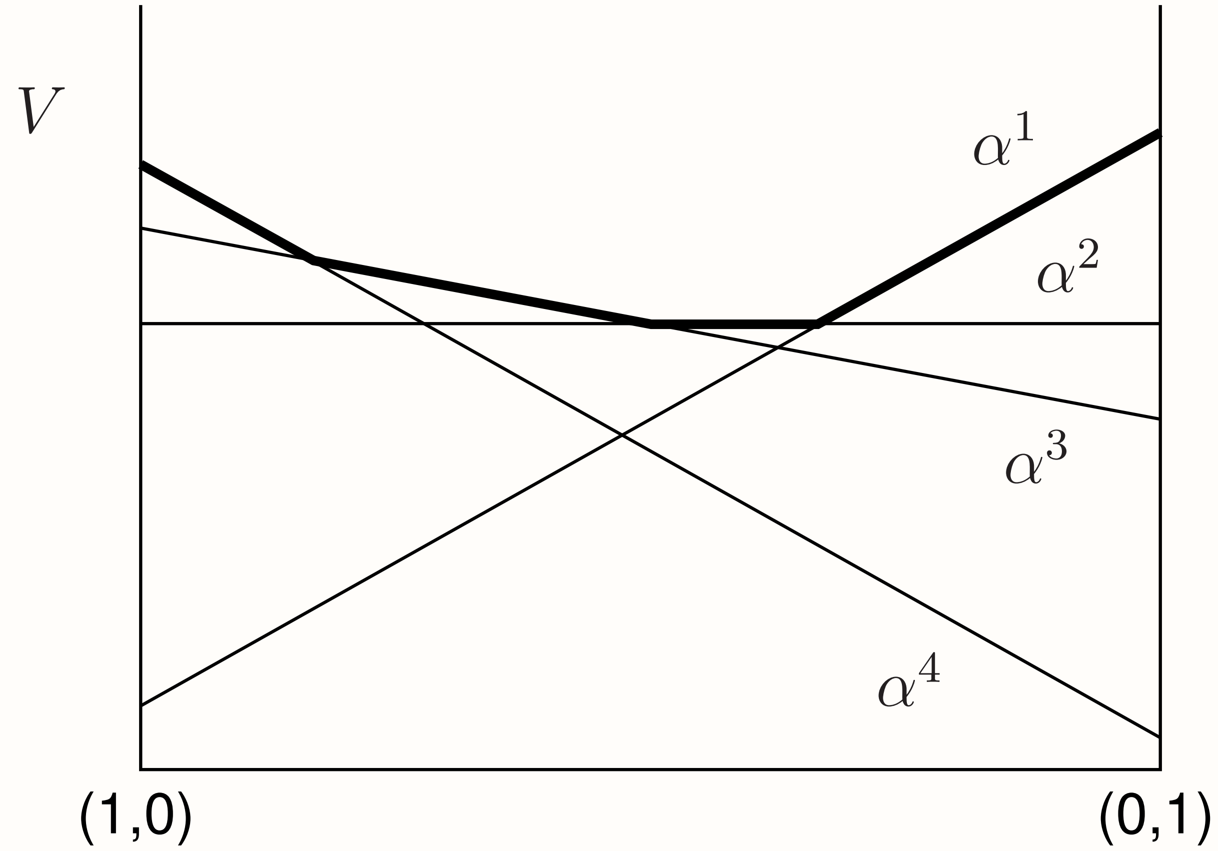
Exact value iteration

Value iteration computes a sequence of value function estimates $V_1, V_2, \ldots, V_n$, using the POMDP backup operator H, $V_{n+1} = HV_n$.

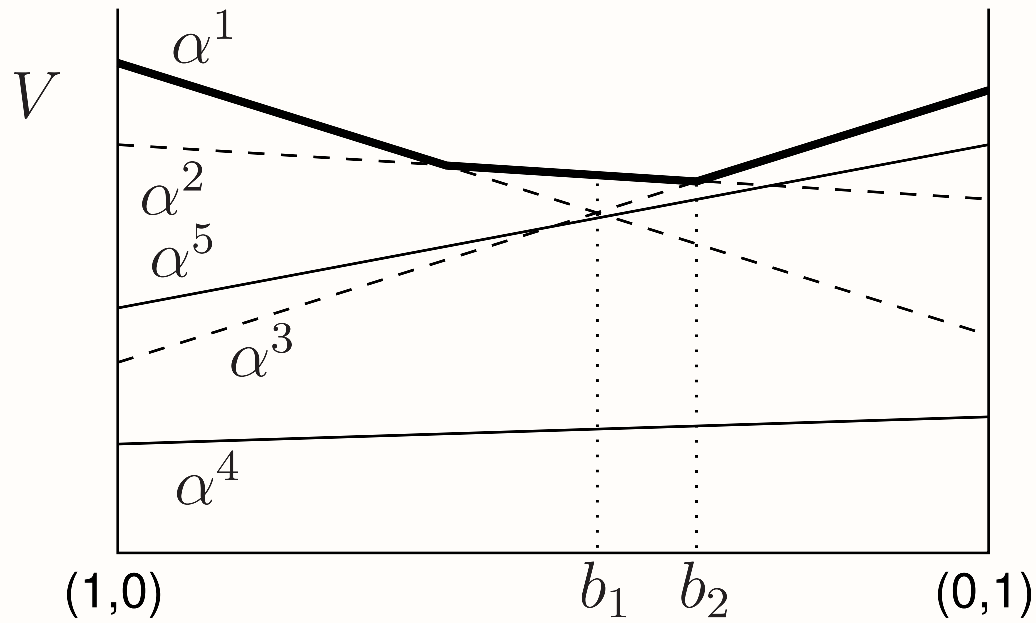


Optimal value functions

The optimal value function of a (finite-horizon) POMDP is piecewise linear and convex: $V(b) = \max_\alpha b \cdot \alpha$.



Vector pruning



Linear program for pruning:

variables: $\forall s \in S, b(s); x$

maximize: x

subject to:

$$b \cdot (\alpha - \alpha') \geq x, \forall \alpha' \in V, \alpha' \neq \alpha$$
$$b \in \Delta(S)$$

Optimal POMDP methods

Enumerate and prune:

- Most straightforward: Monahan (1982)'s enumeration algorithm. Generates a maximum of $|A||V_n|^{|O|}$ vectors at each iteration, hence requires pruning.
- Incremental pruning (Zhang and Liu, 1996; Cassandra et al., 1997).

Search for witness points:

- One Pass (Sondik, 1971; Smallwood and Sondik, 1973).
- Relaxed Region, Linear Support (Cheng, 1988).
- Witness (Cassandra et al., 1994).

Sub-optimal techniques

- Grid-based approximations

  (Drake, 1962; Lovejoy, 1991; Brafman, 1997; Zhou and Hansen, 2001; Bonet, 2002).
- Optimizing finite-state controllers

  (Platzman, 1981; Hansen, 1998b; Poupart and Boutilier, 2004).
- Heuristic search in the belief tree

  (Satia and Lave, 1973; Hansen, 1998a).
- Compression or clustering

  (Roy et al., 2005; Poupart and Boutilier, 2003; Virin et al., 2007).
- Point-based techniques

  (Pineau et al., 2003; Smith and Simmons, 2004; Spaan and Vlassis, 2005; Shani et al., 2007; Kurniawati et al., 2008).
- Monte Carlo tree search

  (Silver and Veness, 2010).

Point-based backup

- For finite horizon V^* is piecewise linear and convex, and for infinite horizons V^* can be approximated arbitrary well by a PWLC value function (Smallwood and Sondik, 1973).

Point-based backup

- For finite horizon V^* is piecewise linear and convex, and for infinite horizons V^* can be approximated arbitrary well by a PWLC value function (Smallwood and Sondik, 1973).
- Given value function V_n and a particular belief point b we can easily compute the vector α_{n+1}^b of HV_n such that

$$\alpha_{n+1}^b = \arg\max_{\{\alpha_{n+1}^k\}_k} b \cdot \alpha_{n+1}^k,$$

where $\{\alpha_{n+1}^k\}_{k=1}^{|HV_n|}$ is the (unknown) set of vectors for HV_n. We will denote this operation $\alpha_{n+1}^b = \texttt{backup}(b)$.

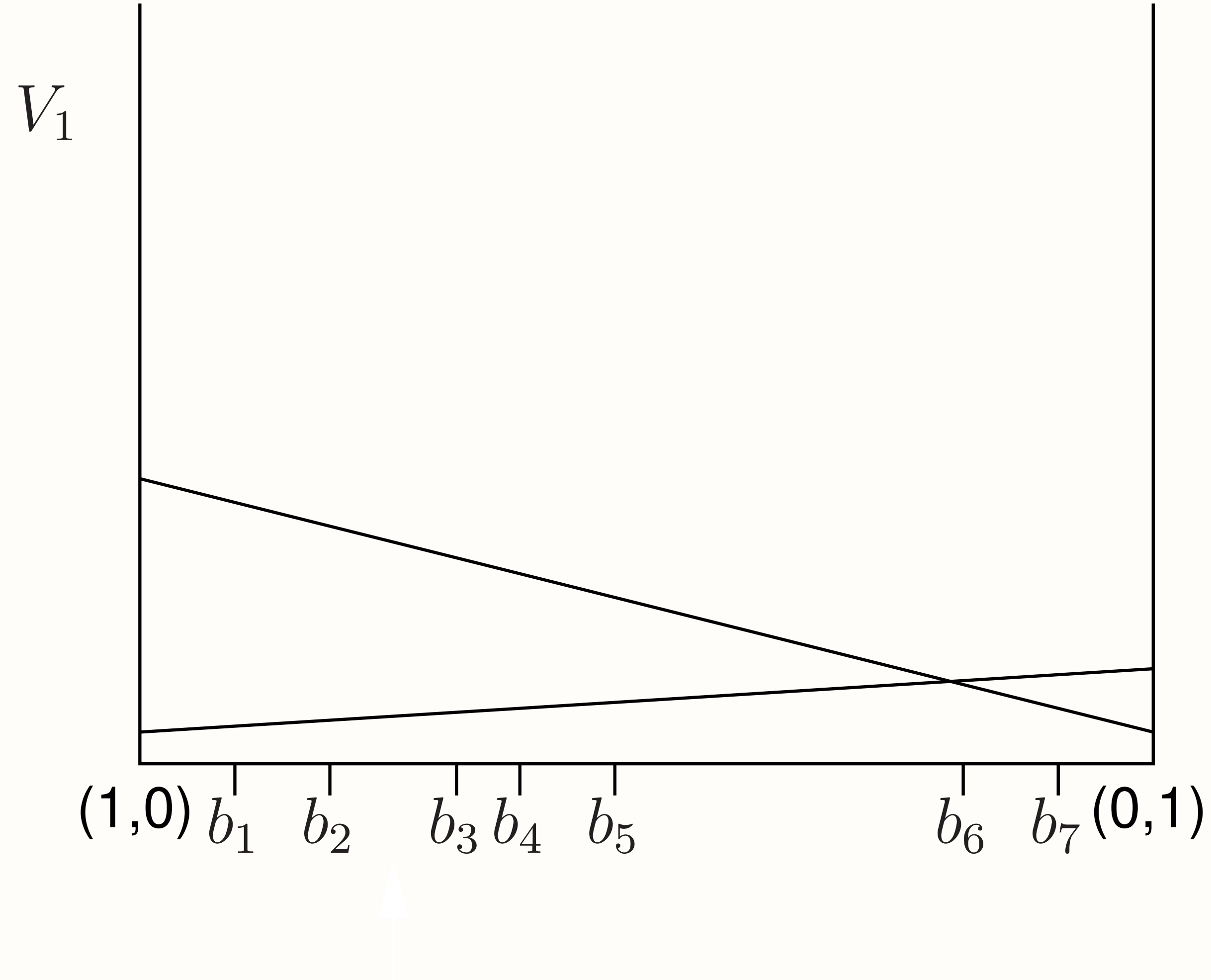
Point-based (approximate) methods

Point-based (approximate) value iteration plans only on a limited set of **reachable** belief points:

1. Let the robot explore the environment.
2. Collect a set B of belief points.
3. Run approximate value iteration on B.

PERSEUS: randomized point-based VI

Idea: at every backup stage improve the value of all $b \in B$.



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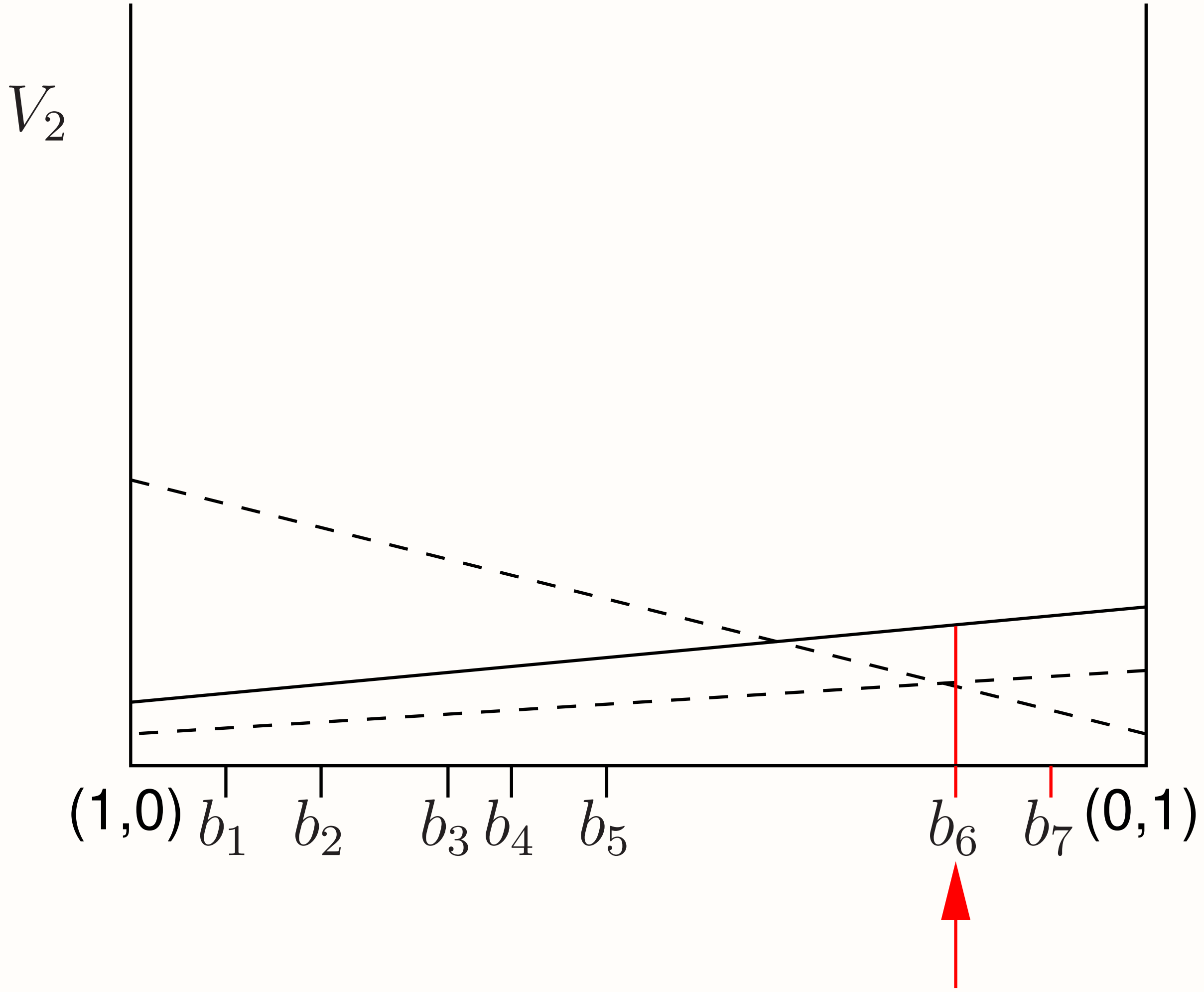

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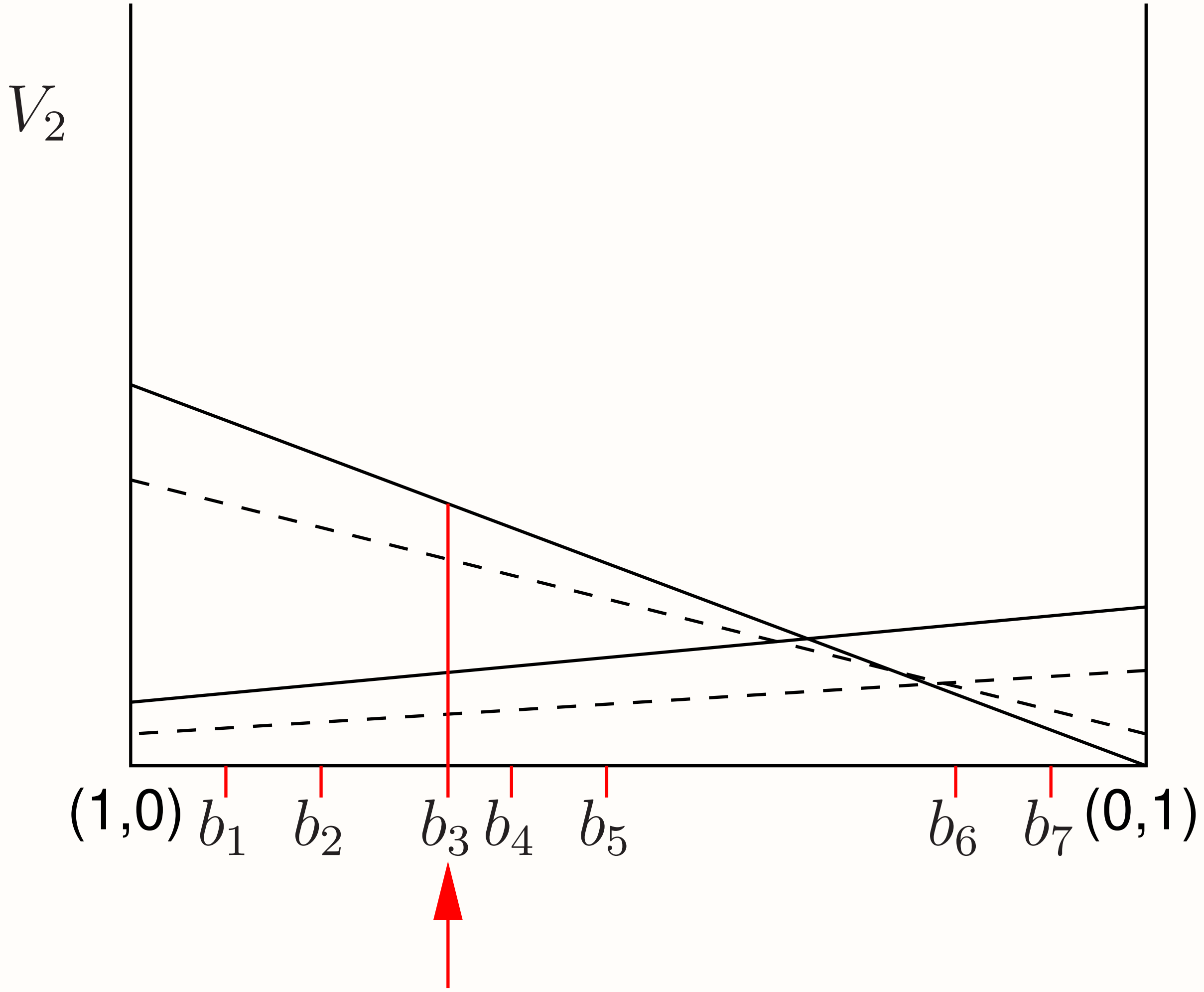

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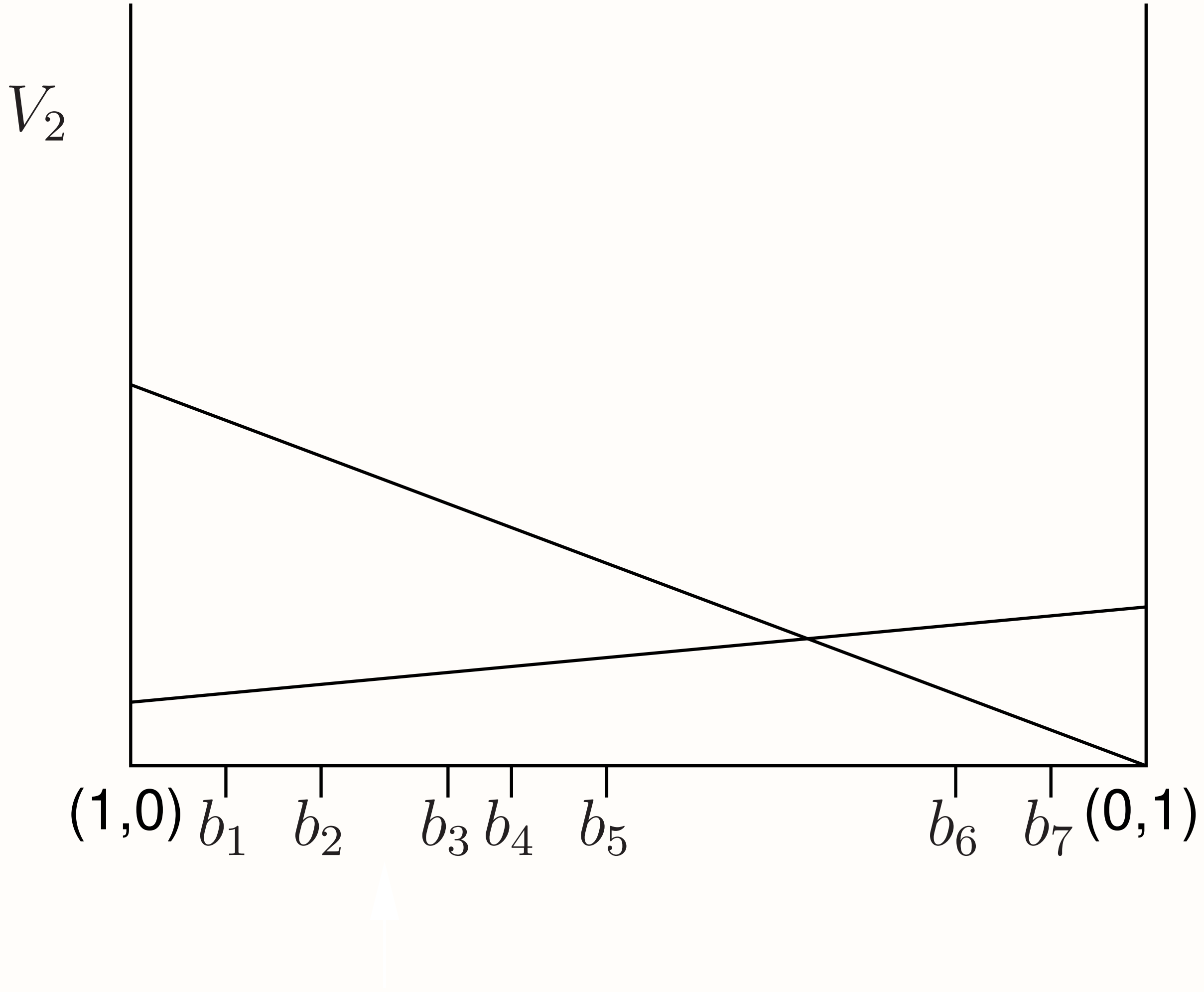

(Spaan and Vlassis, 2005)

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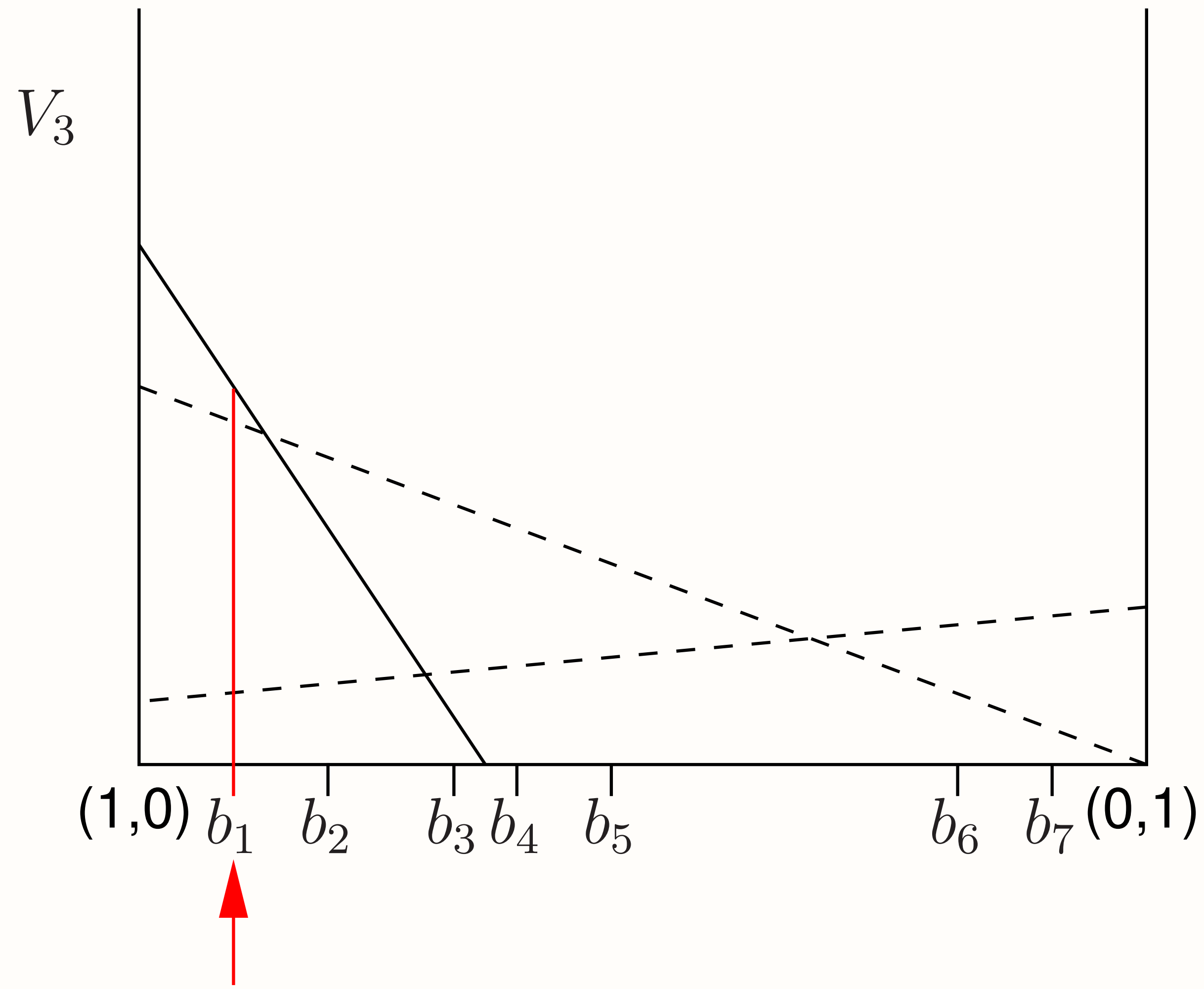

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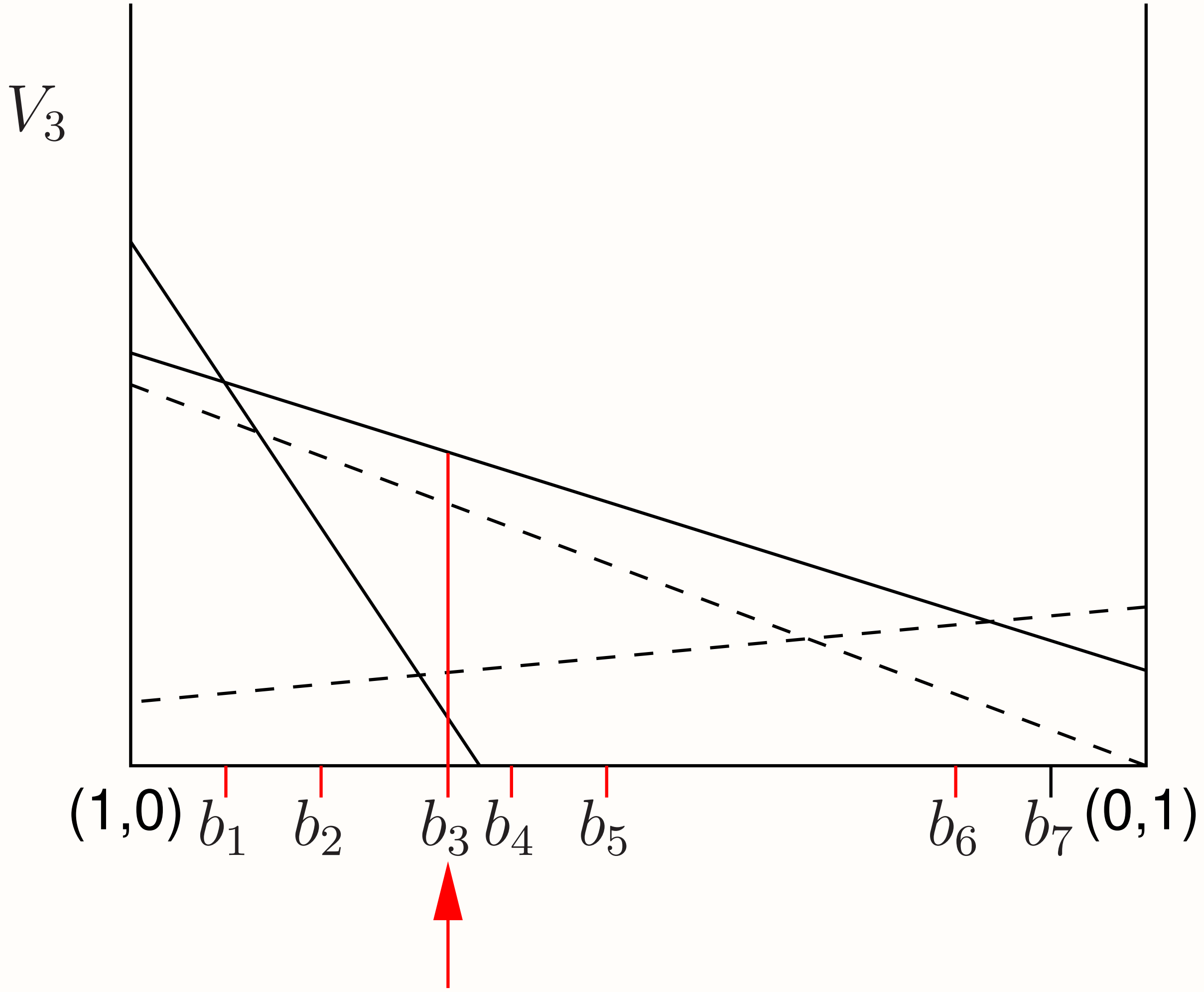

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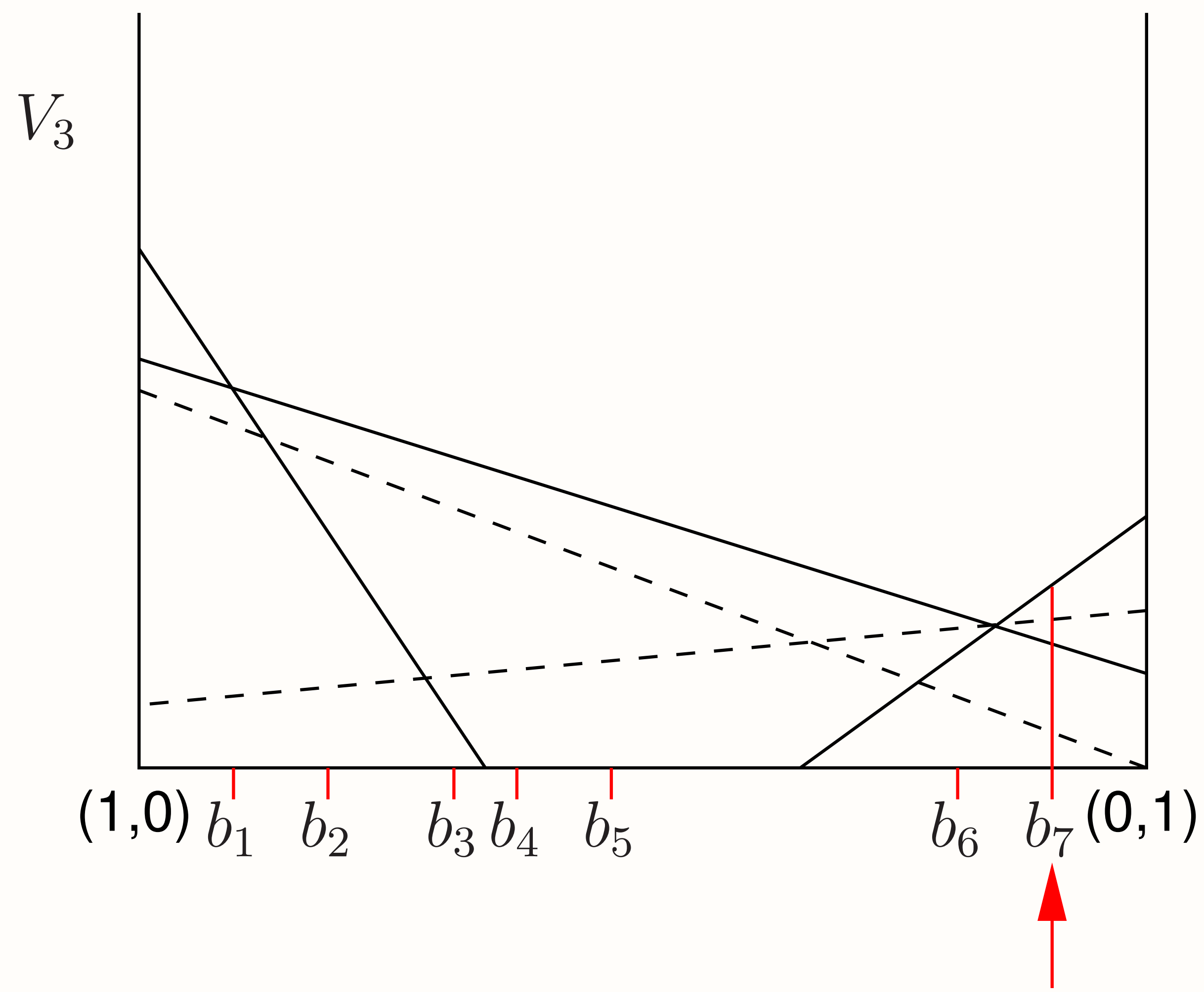

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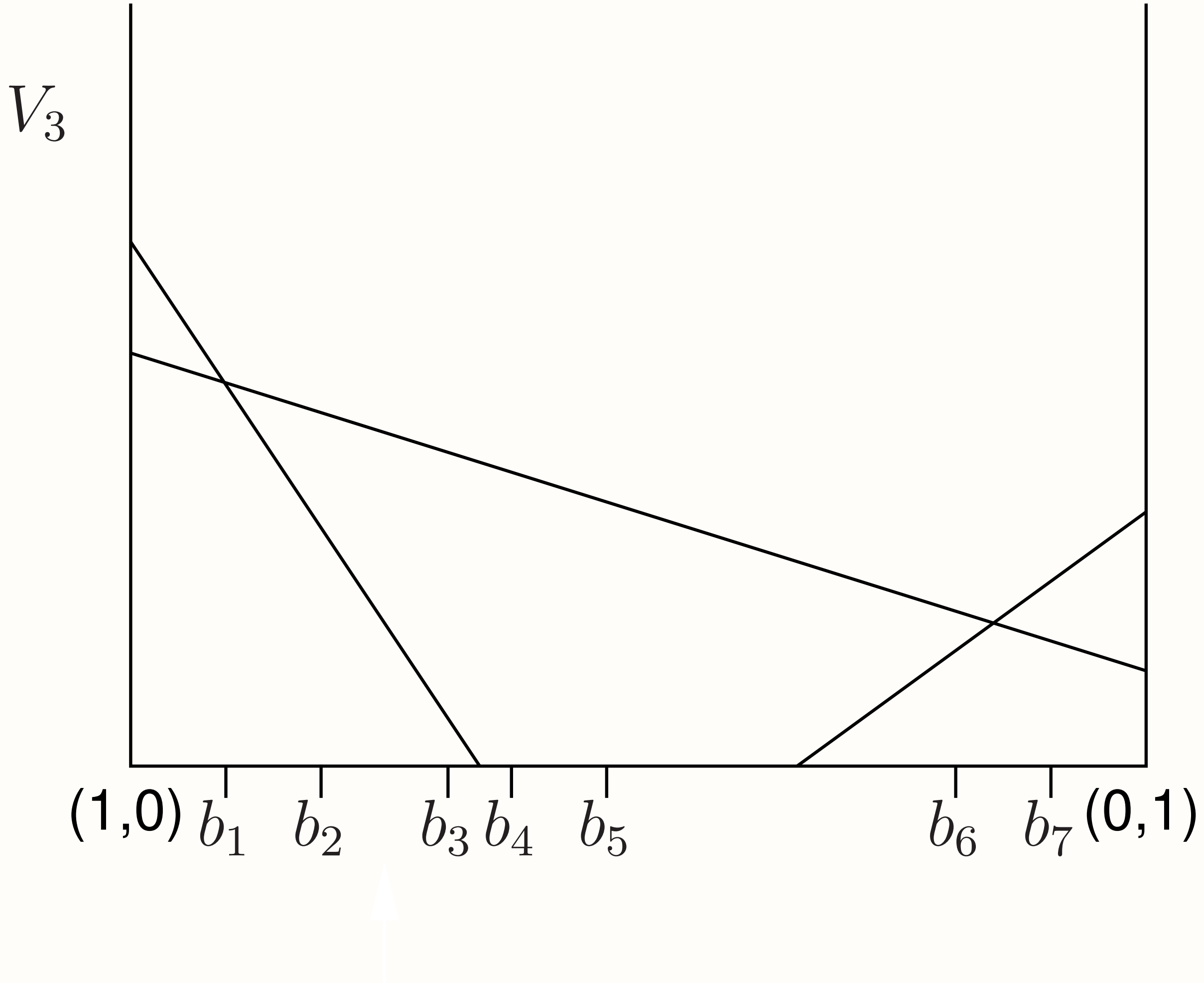

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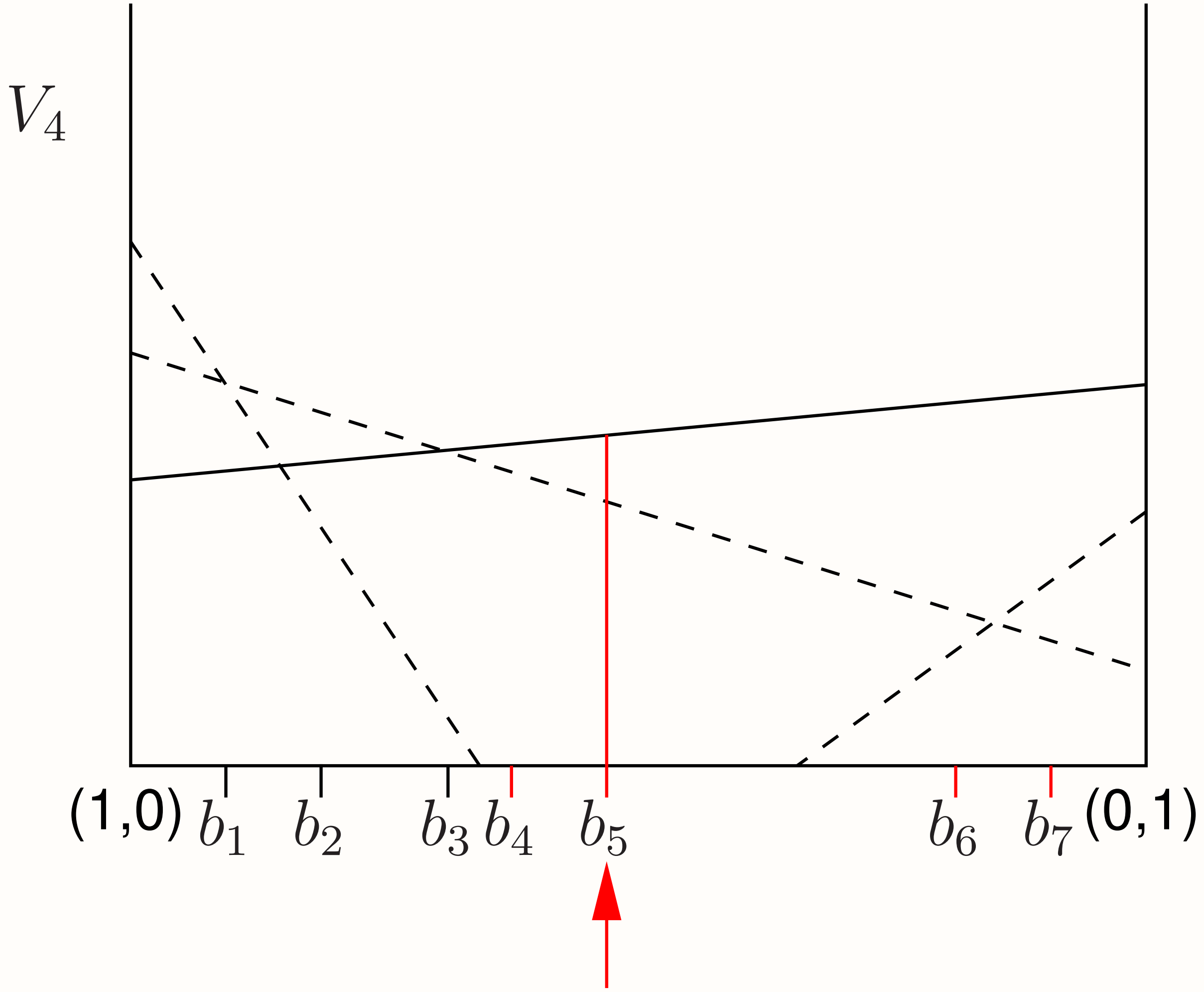

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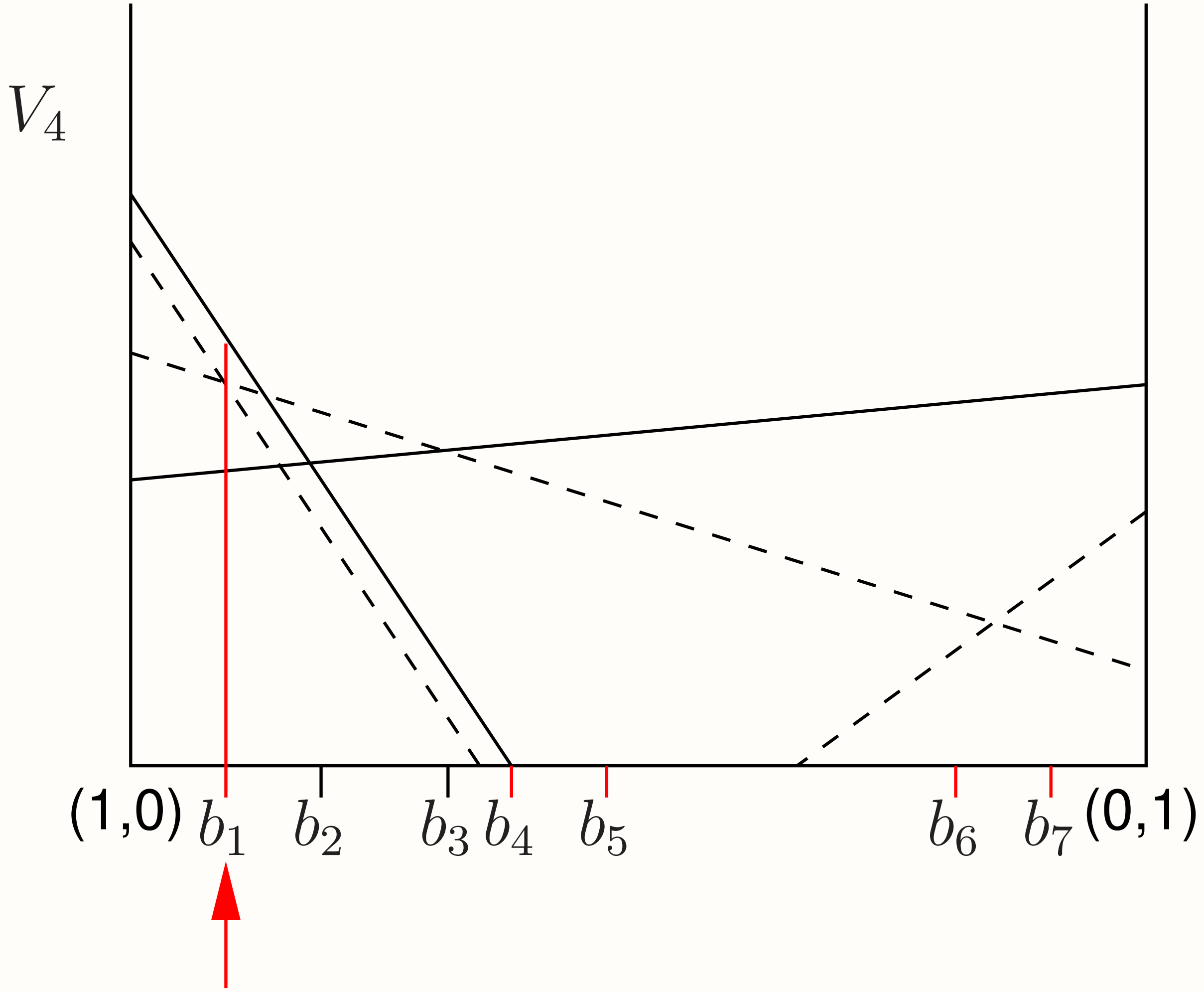

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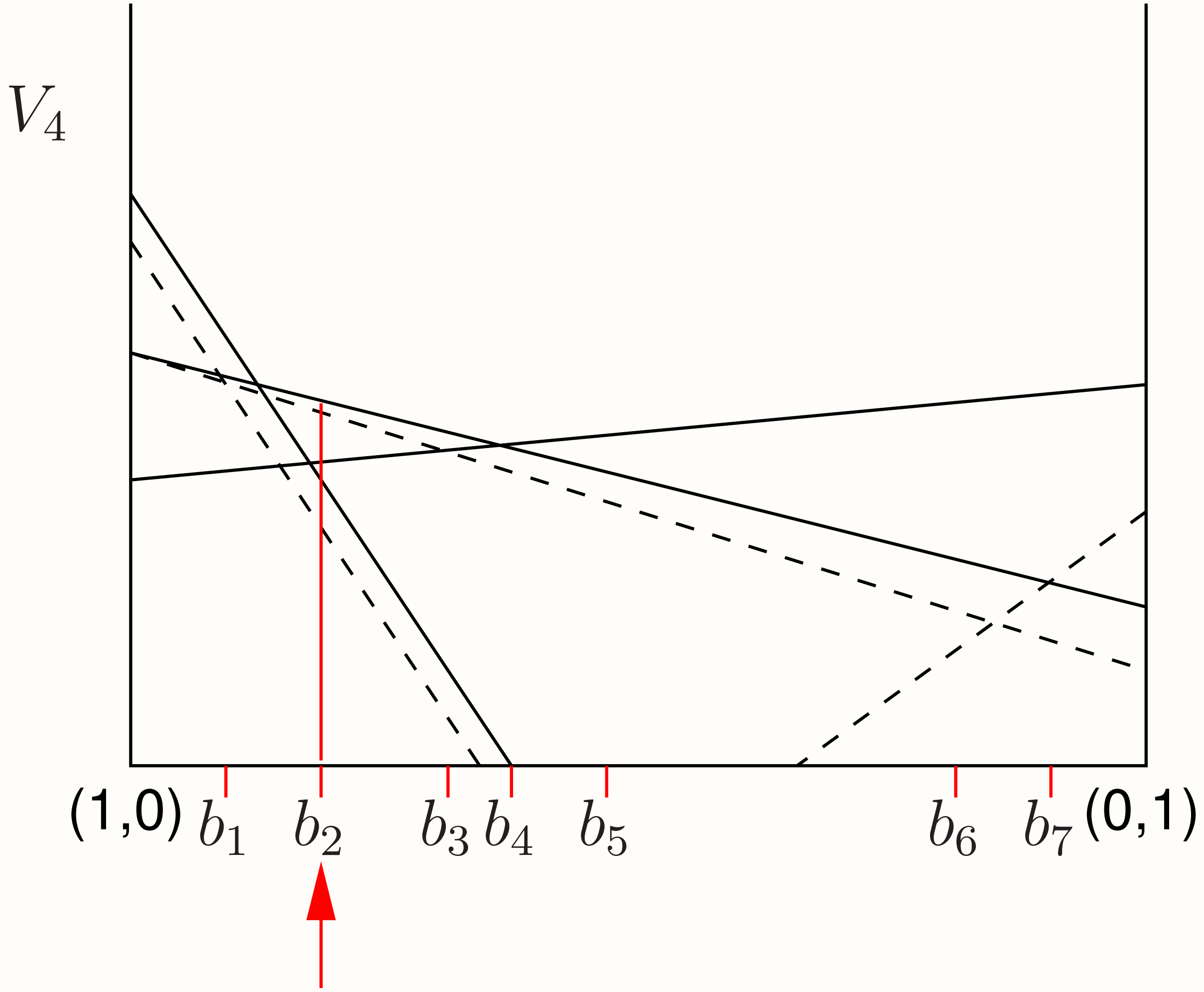

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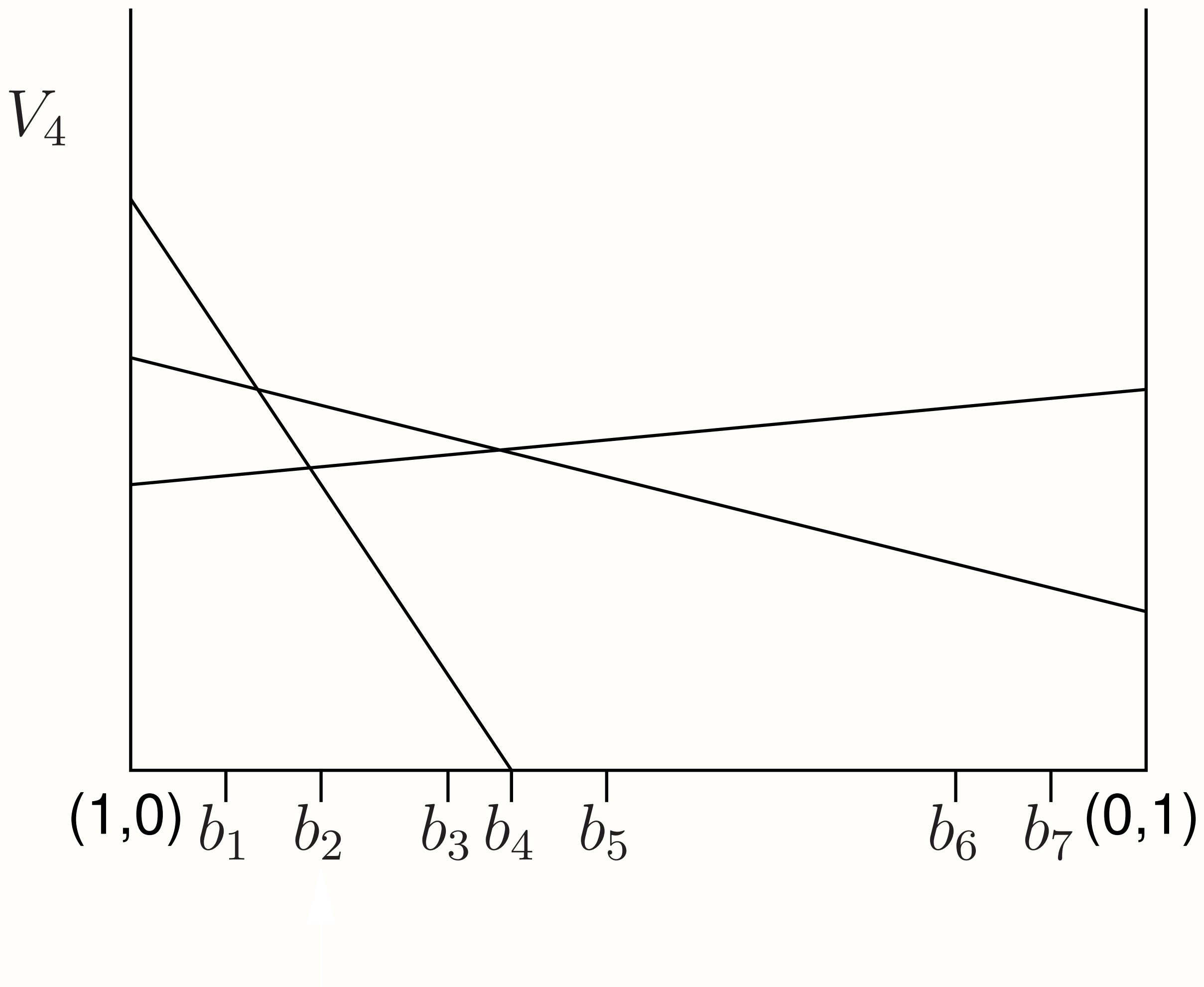


(Spaan and Vlassis, 2005)

High dimensional sensor readings

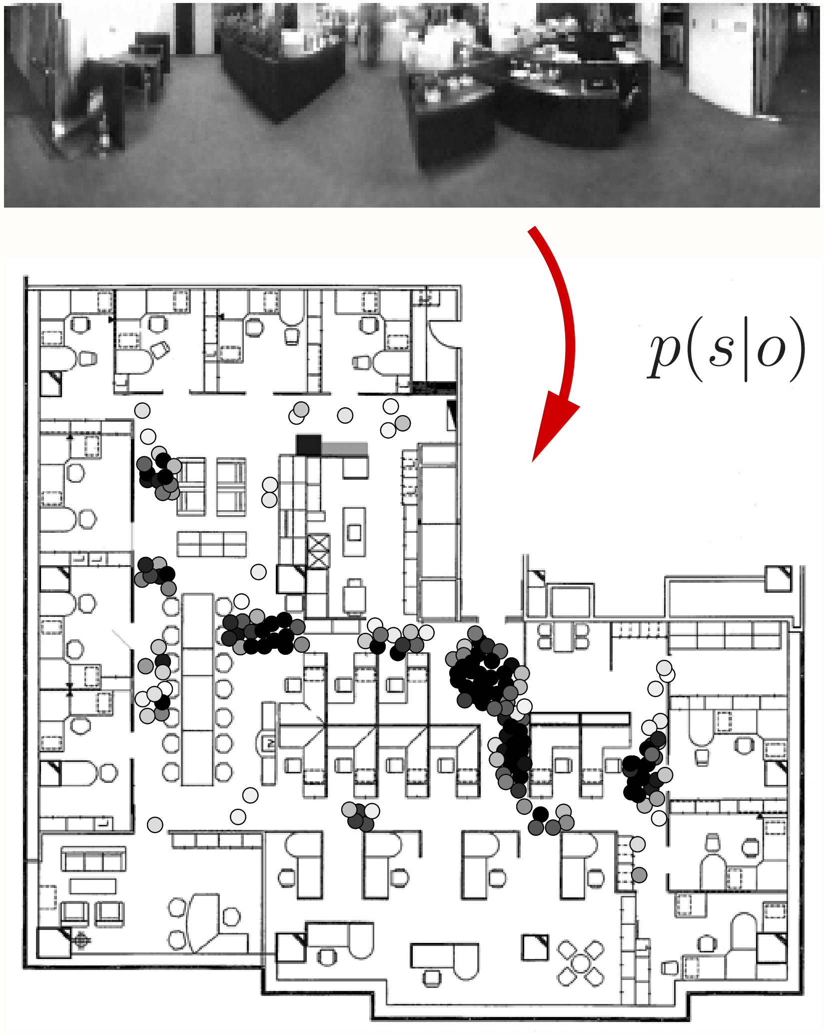
Omnidirectional camera images.

Example images ⇒

Dimension reduction:

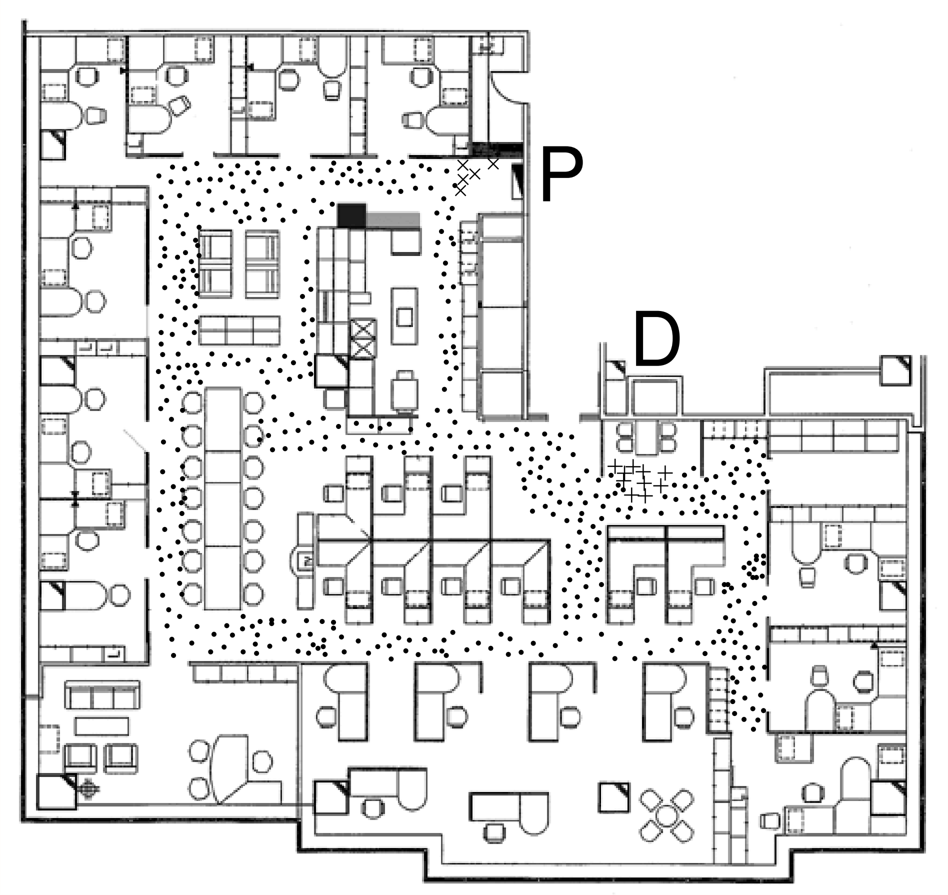
- Collect a database of images and record their location.
- Apply Principal Component Analysis on the image data.
- Project each image to the first 3 eigenvectors, resulting in a 3D feature vector for each image.

Observation model



- We cluster the feature vectors into 10 prototype observations.
- We compute a discrete observation model $p(o|s,a)$ by a histogram operation.

States, actions and rewards



- State: $s = (x, j)$ with x the robot's location and j the mail bit.
- Grid X into 500 locations.
- Actions: $\{\uparrow, \rightarrow, \downarrow, \leftarrow,$ *pickup*, *deliver*$\}$.
- Positive reward: only upon successful mail delivery.

(Spaan and Vlassis, 2004)

Further reading

- Recent book containing chapters on many aspects of decision-theoretic planning (MDPs, POMDPs, Dec-POMDPs):
  - Marco Wiering and Martijn van Otterlo, editors, "Reinforcement Learning: State of the Art", Springer, 2012.

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