

# Exploiting Agent and Type Independence in Collaborative Graphical Bayesian Games

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## Abstract

Efficient collaborative decision making is an important challenge for multiagent systems. Finding optimal joint actions is especially challenging when each agent has only imperfect information about the state of its environment. Such problems can be modeled as *collaborative Bayesian games* in which each agent receives private information in the form of its *type*. However, representing and solving such games requires space and computation time exponential in the number of agents. This article introduces *collaborative graphical Bayesian games (CGBGs)*, which facilitate more efficient collaborative decision making by decomposing the global payoff function as the sum of local payoff functions that depend on only a few agents. We propose a framework for the efficient solution of CGBGs based on the insight that they possess two different types of independence, which we call *agent independence* and *type independence*. In particular, we present a *factor graph* representation that captures both forms of independence and thus enables efficient solutions. In addition, we show how this representation can provide leverage in sequential tasks by using it to construct a novel method for *decentralized partially observable Markov decision processes*. Experimental results in both random and benchmark tasks demonstrate the improved scalability of our methods compared to several existing alternatives.

**keywords:** reasoning under uncertainty, decision-theoretic planning, multiagent decision making, collaborative Bayesian games, decentralized partially observable Markov decision processes

## 1 Introduction

Collaborative multiagent systems are of significant scientific interest, not only because they can tackle inherently distributed problems, but also because they facilitate the decomposition of problems too complex to be tackled by a single agent (Huhns, 1987; Sycara, 1998; Panait and Luke, 2005; Vlassis, 2007; Buşoniu et al., 2008). As a result, a fundamental question in artificial intelligence is how best to design control systems for collaborative multiagent systems. In other words, how should teams of agents act so as to most effectively achieve common goals? When uncertainty and many agents are involved, this question is particularly challenging, and has not yet been answered in a satisfactory way.

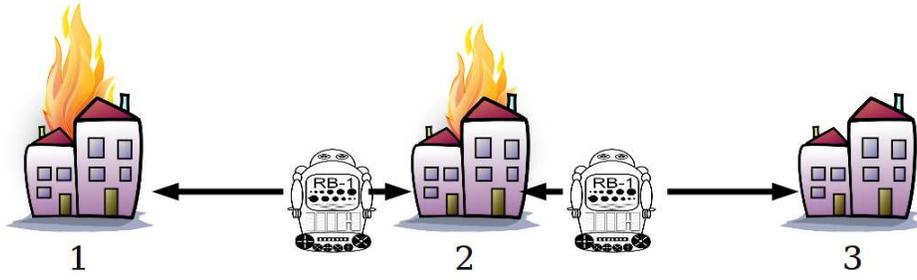


Figure 1: Illustration of multiagent decision making with imperfect information. Both agents are located near house 2 and know that it is on fire. However, each agent receives only a noisy observation of the single neighboring house it can observe in the distance.

A key challenge of collaborative multiagent decision making is the presence of *imperfect information* (Harsanyi, 1967–1968; Kaelbling et al., 1998). Even in single-agent settings, agents may have incomplete knowledge of the state of their environment, e.g., due to noisy sensors. However, in multiagent settings this problem is often greatly exacerbated, as agents have access only to their own sensors, typically a small fraction of those of the complete system. In some cases, imperfect information can be overcome by sharing sensor readings. However, due to bandwidth limitations and synchronization issues, communication-based solutions are often brittle and scale poorly in the number of agents.

As an example, consider the situation depicted in Fig. 1. After an emergency call by the owner of house 2, two firefighting agents are dispatched to fight the fire. While each agent knows there is a fire at house 2, the agents are not sure whether fire has spread to the neighboring houses. Each agent can potentially observe flames at one of the neighboring houses (agent 1 observes house 1 and agent 2 observes house 3) but neither has perfect information about the state of the houses. As a result, effective decision making is difficult. If agent 1 observes flames in house 1, it may be tempted to fight fire there rather than at house 2. However, the efficacy of doing so depends on whether agent 2 will stay to fight fire in house 2, which in turn depends on whether agent 2 observes flames in house 3, a fact unknown to agent 1.

*Strategic games*, the traditional models of game theory, are poorly suited to modeling such problems because they assume that there is only one state, which is known to all the agents. In contrast, in the example of Fig. 1, each agent has only a partial view of the state, i.e., from each agent’s individual perspective, multiple states are possible. Such problems of multiagent decision making with imperfect information can be modeled with *Bayesian games (BGs)* (Harsanyi, 1967–1968; Osborne and Rubinstein, 1994). In a BG, each agent has a *type* that specifies what private information it holds. For example, an agent’s type may correspond to an observation that it makes but the other agents do not. Before the agents select actions, their types are drawn from a distribution. Then, the payoffs they receive depend not only on the actions they choose, but also on their types. Problems in which the agents have a common goal can be modeled as *collaborative Bayesian games (CBGs)*, in which all agents share a single global payoff function. Unfortunately, solving CBGs efficiently is difficult, as both the space needed to represent the payoff function and the computation time needed to find optimal joint actions scale exponentially with the number of agents.

In this article, we introduce *collaborative graphical Bayesian games (CGBGs)*, a new framework designed to facilitate more efficient collaborative decision making with imperfect information. As in strategic games (Guestrin et al., 2002; Kok and Vlassis, 2006), global payoff functions in Bayesian games can often be decomposed as the sum of *local payoff functions*, each of which depends on the actions of only a few agents. We call such games graphical because this decomposition can be expressed as an *interaction hypergraph* that specifies which agents participate in which local payoff functions.

Our main contribution is to demonstrate how this graphical structure can be exploited to solve CGBGs more efficiently. Our approach is based on the critical insight that CGBGs contain two fundamentally different types of independence. Like graphical strategic games, CGBGs possess *agent independence*: each local payoff function depends on only a subset of the agents. However, we identify that GBGs also possess *type independence*: since only one type per agent is realized in a given game, the expected payoff decomposes as the sum of *contributions* that depend on only a subset of types.

We propose a *factor graph* representation that captures both agent and type independence. Then, we show how such a factor graph can be used to find optimal joint policies via *nonserial dynamic programming* (Bertele and Brioschi, 1972; Guestrin et al., 2002). While this approach is faster than a naïve alternative, we prove that its computational complexity remains exponential in the number of types. However, we also show how the same factor graph facilitates even more efficient, scalable computation of approximate solutions via MAX-PLUS (Kok and Vlassis, 2006), a message-passing algorithm. In particular, we prove that each iteration of max-plus is tractable for small local neighborhoods.

Alternative solution approaches for CGBGs can be found among existing techniques. For example, a CGBG can be converted to a multiagent influence diagram (MAID) (Koller and Milch, 2003). However, since the resulting MAID has a single strongly connected component, the divide and conquer technique proposed by Koller and Milch reduces to brute-force search. Another approach is to convert CGBGs to non-collaborative graphical strategic games, for which efficient solution algorithms exist (Vickrey and Koller, 2002; Ortiz and Kearns, 2003; Daskalakis and Papadimitriou, 2006). However, the conversion process essentially strips away the CGBG’s type independence, resulting in an exponential increase in the worst-case size of the payoff function. CGBGs can also be modeled as constraint optimization problems (Modi et al., 2005), for which some methods implicitly exploit type independence (Oliehoek et al., 2010; Kumar and Zilberstein, 2010). However, these methods do not explicitly identify type independence and do not exploit agent independence.

Thus, the key advantage of the approach presented in this article is the simultaneous exploitation of both agent and type independence. We present a range of experimental results that demonstrate that this advantage leads to better scalability than several alternatives with respect to the number of agents, actions, and types.

While CGBGs model an important class of collaborative decision-making problems, they apply only to *one-shot* settings, i.e., each agent needs to select only one action. However, CGBG solution methods can also provide substantial leverage in *sequential* tasks, in which agents take a series of actions over time. We illustrate the benefits of CGBGs in such settings by using them to construct a novel method for solving *decentralized partially observable Markov decision processes (Dec-POMDPs)* (Bernstein et al., 2002). Our method extends an existing approach in which each stage of the Dec-POMDP is modeled as a CBG. In particular, we show how approximate inference and factored value functions can be used to reduce the problem to a set of CGBGs, which can be solved using our novel approach. Additional experiments in multiple Dec-POMDP benchmark tasks demonstrate better scalability in the number of agents than several alternative methods. In particular, for a sequential version of a firefighting task as described above, we were able to scale to 1000 agents, where previous approaches to Dec-POMDPs have not been demonstrated beyond 20 agents.

The rest of this paper is organized as follows. Sec. 2 provides background by introducing collaborative (Bayesian) games and their solution methods. Sec. 3 introduces CGBGs, which capture both agent and type independence. This section also presents solution methods that exploit such independence, analyzes their computational complexity, and empirically evaluates their performance. In Sec. 4, we show that the impact of our work extends to sequential tasks by presenting and evaluating a new Dec-POMDP method based on CGBGs. Sec. 5 discusses related work, Sec. 6 discusses possible directions for future work, and Sec. 7 concludes.

## 2 Background

In this section, we provide background on various game-theoretic models for collaborative decision making. We start with the well-known framework of *strategic games* and discuss their graphical counterparts, which allow for compact representations of problems with many agents. Next, we discuss *Bayesian games*, which take into account different private information available to each agent. These models provide a foundation for understanding *collaborative graphical Bayesian games*, the framework we propose in Section 3.

### 2.1 Strategic Games

The *strategic game* (SG) framework (Osborne and Rubinstein, 1994) is probably the most studied of all game-theoretic models. Strategic games are also called *normal form games* or *matrix games*, since two-agent games can be represented by matrices. We first introduce the formal model and then discuss solution methods and compact representations.

#### 2.1.1 The Strategic Game Model

In a strategic game, a set of agents participate in a one-shot interaction in which they each select an action. The outcome of the game is determined by the combination of selected actions, which leads to a payoff for each agent.

**Definition 2.1.** A *strategic game* (SG) is a tuple  $\langle \mathcal{D}, \mathcal{A}, \langle u_1, \dots, u_n \rangle \rangle$ , where

- $\mathcal{D} = \{1, \dots, n\}$  is the set of  $n$  agents,
- $\mathcal{A} = \times_i \mathcal{A}_i$  is the set of joint actions  $\mathbf{a} = \langle a_1, \dots, a_n \rangle$ ,
- $u_i : \mathcal{A} \rightarrow \mathbb{R}$  is the payoff function of agent  $i$ .

This article focuses on collaborative decision making: settings in which the agents have the same goal, which is modeled by the fact that the payoffs the agents receive are identical.

**Definition 2.2.** A *collaborative strategic game* (CSG) is a strategic game in which each agent has the same payoff function:  $\forall_{i,j} \forall_{\mathbf{a}} u_i(\mathbf{a}) = u_j(\mathbf{a})$ .

In the collaborative case, we drop the subscript on the payoff function and simply write  $u$ . CSGs are also called *identical payoff games* or *team games*.

#### 2.1.2 Solution Concepts

A solution to an SG is a description of what actions each agent should take. While many solution concepts have been proposed, one of central importance is the equilibrium introduced by Nash (1950).

**Definition 2.3.** A joint action  $\mathbf{a} = \langle a_1, \dots, a_i, \dots, a_n \rangle$  is a *Nash equilibrium* (NE) if and only if

$$u_i(\langle a_1, \dots, a_i, \dots, a_n \rangle) \geq u_i(\langle a_1, \dots, a'_i, \dots, a_n \rangle), \quad \forall_{i \in \mathcal{D}}, \forall_{a'_i \in \mathcal{A}_i}. \quad (2.1)$$

Intuitively, an NE is a joint action such that no agent can improve its payoff by changing its own action. A game may have zero, one or multiple NEs.<sup>1</sup> When there are multiple NEs, the concept of *Pareto optimality* can help distinguish between them.

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<sup>1</sup>Nash proved that every finite game contains at least one NE if actions are allowed to be played with a particular probability, i.e., if *mixed strategies* are allowed.

**Definition 2.4.** A joint action  $\mathbf{a}$  is *Pareto optimal* if there is no other joint action  $\mathbf{a}'$  that specifies at least the same payoff for every agent and a higher payoff for at least one agent, i.e., there exists no  $\mathbf{a}'$  such that

$$\forall_i u_i(\mathbf{a}') \geq u_i(\mathbf{a}) \quad \wedge \quad \exists_i u_i(\mathbf{a}') > u_i(\mathbf{a}). \quad (2.2)$$

If there does exist an  $\mathbf{a}'$  such that (2.2) holds, then  $\mathbf{a}'$  *Pareto dominates*  $\mathbf{a}$ .

**Definition 2.5.** A joint action  $\mathbf{a}$  is a *Pareto-optimal Nash equilibrium (PONE)* if and only if it is an NE and there is no other  $\mathbf{a}'$  such that  $\mathbf{a}'$  is an NE and Pareto dominates  $\mathbf{a}$ .

Note that this definition does not require that  $\mathbf{a}$  is Pareto optimal. On the contrary, there may exist an  $\mathbf{a}'$  that dominates  $\mathbf{a}$  but is not an NE.

### 2.1.3 Solving CSGs

In collaborative strategic games, each maximizing entry of the payoff function is a PONE. Therefore, finding a PONE requires only looping over all the entries in  $u$  and selecting a maximizing one, which takes time linear in the size of the game. However, coordination issues can arise when searching for a PONE with a decentralized algorithm, e.g., when there are multiple maxima. Ensuring that the agents select the same PONE can be accomplished by imposing certain social conventions or through repeated interactions (Boutilier, 1996). In this article, we assume that the game is solved in an off-line *centralized* planning phase and that the joint strategy is then distributed to the agents, who merely execute the actions in the on-line phase. We focus on the design of cooperative teams of agents, for which this is a reasonable assumption.

## 2.2 Collaborative Graphical Strategic Games

Although CSGs are conceptually easy to solve, the game description scales exponentially with the number of agents. That is, the size of the payoff function and thus the time required for the trivial algorithm is  $O(|\mathcal{A}_*|^n)$ , where  $|\mathcal{A}_*|$  denotes the size of the largest individual action set. This is a major obstacle in the representation and solution of SGs for large values of  $n$ . Many games, however, possess independence because not all agents need to coordinate directly (Guestrin et al., 2002; Kearns et al., 2001; Kok and Vlassis, 2006). This idea is formalized by collaborative graphical strategic games.

### 2.2.1 The Collaborative Graphical SG Model

In collaborative graphical SGs, the payoff function is decomposed into *local payoff functions*, each having limited *scope*, i.e., only subsets of agents participate in each local payoff function.

**Definition 2.6.** A *collaborative graphical strategic game (CGSG)* is a CSG whose payoff function  $u$  decomposes over a number  $\rho$  of *local payoff functions*  $\mathcal{U} = \{u^1, \dots, u^\rho\}$ :

$$u(\mathbf{a}) = \sum_{e=1}^{\rho} u^e(\mathbf{a}_e). \quad (2.3)$$

Each local payoff function  $u^e$  has scope  $\mathbb{A}(u^e)$ , the subset of agents that participate in  $u^e$ . Here  $\mathbf{a}_e$  denotes the *local joint action*, i.e., the profile of actions of the agents in  $\mathbb{A}(u^e)$ .

Each local payoff component can be interpreted as a hyper-edge in an *interaction hyper-graph*  $IG = \langle \mathcal{D}, \mathcal{E} \rangle$  in which the nodes  $\mathcal{D}$  are agents and the hyper-edges  $\mathcal{E}$  are local payoff functions (Nair et al., 2005; Oliehoek et al., 2008c). Two (or more) agents are connected by such a (hyper-)edge  $e \in \mathcal{E}$  if and only if they

participate in the corresponding local payoff function  $u^e$ .<sup>2</sup> Note that we shall abuse notation in that  $e$  is used as an index into the set of local payoff functions and as an element of the set of scopes. Fig. 2a shows the interaction hyper-graph of a five-agent CGSG. If only two agents participate in each local payoff function, the interaction hyper-graph reduces to a regular graph and the framework is identical to that of *coordination graphs* (Guestrin et al., 2002; Kok and Vlassis, 2006).

CGSGs are also similar to *graphical games* (Kearns et al., 2001; Kearns, 2007; Soni et al., 2007). However, there is a crucial difference in the meaning of the term ‘graphical’. In CGSGs, it indicates that the single, common payoff function ( $u = u_1 = \dots = u_n$ ) decomposes into local payoff functions, each involving subsets of agents. However, all agents participate in the common payoff function (otherwise they would be irrelevant to the game). In contrast, graphical games are typically not collaborative. Thus, in that context, the term indicates that the *individual* payoff functions  $u_1, \dots, u_n$  involve subsets of agents. However, these individual payoff functions do not decompose into sums of local payoff functions.

### 2.2.2 Solving CGSGs

Solving a collaborative graphical strategic game entails finding a maximizing joint action. However, if the representation of a particular problem is *compact*, i.e. exponentially smaller than its non-graphical (i.e., CSG) representation, then the trivial algorithm of Sec. 2.1.3 runs in exponential time. *Non-serial dynamic programming (NDP)* (Bertele and Brioschi, 1972), also known as *variable elimination* (Guestrin et al., 2002; Vlassis, 2007) and *bucket elimination* (Dechter, 1999), can find an optimal solution much faster by exploiting the structure of the problem. We will explain NDP in more detail in Sec. 3.4.1.

Alternatively, MAX-PLUS, a message-passing algorithm described further in Sec. 3.4.2, can be applied to the interaction graph (Kok and Vlassis, 2005, 2006). In practice, MAX-PLUS is often much faster than NDP (Kok and Vlassis, 2005, 2006; Farinelli et al., 2008; Kim et al., 2010). However, when more than two agents participate in the same hyper-edge (i.e., when the interaction graph is a hyper-graph), message passing cannot be conducted on the hyper-graph itself. Fortunately, an interaction hyper-graph can be translated into a *factor graph* (Kschischang et al., 2001; Loeliger, 2004) to which MAX-PLUS is applicable. The resulting factor graph is a bipartite graph containing one set of nodes for all the local payoff functions and another for all the agents.<sup>3</sup> A local payoff function  $u^e$  is connected to an agent  $i$  if and only if  $i \in \mathbb{A}(u^e)$ . Fig. 2 illustrates the relationship between an interaction hyper-graph and a factor graph.

It is also possible to convert a CGSG into a (non-collaborative) graphical SG by combining all payoff functions in which an agent participates into one normalized, individual payoff function.<sup>4</sup> Several methods for solving graphical SGs are then applicable (Vickrey and Koller, 2002; Ortiz and Kearns, 2003; Daskalakis and Papadimitriou, 2006). Unfortunately, the individual payoff functions resulting from this transformation are exponentially larger in the worst case.

## 2.3 Bayesian Games

Although strategic games provide a rich model of interactions between agents, they assume that each agent has complete knowledge of all relevant information and can therefore perfectly predict the payoffs that result from each joint action. As such, they cannot explicitly represent cases where agents possess private information that influences the effects of actions. For example, in the firefighting example depicted in Fig. 1, there is no natural way in a strategic game to represent the fact that each agent has different information about the state

<sup>2</sup>This constitutes an *edge-based decomposition*, which stands in contrast to *agent-based decompositions* (Kok and Vlassis, 2006). We focus on edge-based decompositions because they are more general.

<sup>3</sup>In the terminology of factor graphs, the local payoff functions correspond to factors and the agents to variables whose domains are the agents’ actions.

<sup>4</sup>This corresponds to converting an edge-based representation to an agent-based representation (Kok and Vlassis, 2006).

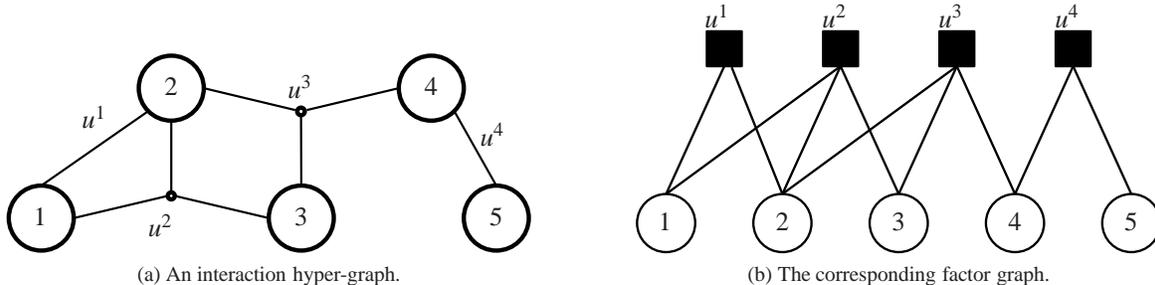


Figure 2: A CGSG with five agents. In (a), each node is an agent and each hyper-edge is a local payoff function. In (b), the circular nodes are agents and the square nodes are local payoff functions, with edges indicating in which local payoff function each agent participates.

of the houses. In more complex problems with a large number of agents, modeling private information is even more important, since assuming that so many agents have perfect knowledge of the complete state of a complex environment is rarely realistic. In this section, we describe *Bayesian games*, which augment the strategic game framework to explicitly model private information. As before, we focus on the collaborative case.

### 2.3.1 The Bayesian Game Model

A Bayesian game, also called a *strategic game of imperfect information* (Osborne and Rubinstein, 1994) is an augmented strategic game in which the players hold private information. The private information of agent  $i$  defines its *type*  $\theta_i \in \Theta_i$ . The payoffs the agents receive depend not only on their actions, but also on their types. Formally, a Bayesian game is defined as follows:

**Definition 2.7.** A *Bayesian game (BG)* is a tuple  $\langle \mathcal{D}, \mathcal{A}, \Theta, \Pr(\Theta), \langle u_1, \dots, u_n \rangle \rangle$ , where

- $\mathcal{D}, \mathcal{A}$  are the sets of agents and joint actions as in an SG,
- $\Theta = \times_{i \in \mathcal{D}} \Theta_i$  is the set of joint types  $\theta = \langle \theta_1, \dots, \theta_n \rangle$ ,
- $\Pr(\Theta)$  is the distribution over joint types, and
- $u_i : \Theta \times \mathcal{A} \rightarrow \mathbb{R}$  is the payoff function of agent  $i$ .

In many problems, the types are a probabilistic function of a hidden state, i.e., based on a hidden state, there is some probability  $\Pr(\theta)$  for each joint type. This is typically the case, as in the example below, when an agent’s type corresponds to a private observation it makes about such a state. However, this hidden state is not a necessary component of a BG. On the contrary, BGs can also model problems where the types correspond to intrinsic properties of the agents. For instance, in a employee recruitment game, a potential employee’s type could correspond to whether or not he or she is a hard worker.

**Definition 2.8.** A *collaborative Bayesian game (CBG)* is a Bayesian game with identical payoffs:

$$\forall_{i,j} \forall_{\theta} \forall_{\mathbf{a}} u_i(\theta, \mathbf{a}) = u_j(\theta, \mathbf{a}).$$

In a strategic game, the agents simply select actions. However, in a BG, the agents can condition their actions on their types. Consequently, agents in BGs select policies instead of actions. A joint policy  $\beta = \langle \beta_1, \dots, \beta_n \rangle$ , consists of individual policies  $\beta_i$  for each agent  $i$ . Deterministic (pure) individual policies are mappings from types to actions  $\beta_i : \Theta_i \rightarrow \mathcal{A}_i$ , while stochastic policies map each type  $\theta_i$  to a probability distribution over actions  $\Pr(\mathcal{A}_i)$ .

state $s$	$\Pr(s)$	$\Pr(\theta s)$			
		$\langle F_1, F_2 \rangle$	$\langle F_1, N_2 \rangle$	$\langle N_1, F_2 \rangle$	$\langle N_1, N_2 \rangle$
no neighbors on fire	0.7	0.01	0.09	0.09	0.81
house 1 on fire	0.10	0.09	0.81	0.01	0.09
house 3 on fire	0.15	0.09	0.01	0.81	0.09
both on fire	0.05	0.81	0.09	0.09	0.01
	$\Pr(\theta)$	0.07	0.15	0.19	0.59

Table 1: The conditional probabilities of the joint types given states and the resulting distribution over joint types for the two-agent firefighting problem.

state $s$	Payoff of joint actions $u(s, \mathbf{a})$			
	$\langle H_1, H_2 \rangle$	$\langle H_1, H_3 \rangle$	$\langle H_2, H_2 \rangle$	$\langle H_2, H_3 \rangle$
no neighbors on fire	+2	0	+3	+2
house 1 on fire	+4	+2	+3	+2
house 3 on fire	+2	+2	+3	+4
both on fire	+4	+4	+3	+4

Table 2: The payoffs as a function of the joint actions and hidden state for the two-agent firefighting problem.

### Example: Two-Agent Fire Fighting

As an example, consider a formal model of the situation depicted in Fig. 1. The agents each have two actions available: agent 1 can fight fire at the first two houses ( $H_1$  and  $H_2$ ) and agent 2 can fight fire at the last two houses ( $H_2$  and  $H_3$ ). Both agents are located near  $H_2$  and therefore know whether it is burning. However, they are uncertain whether  $H_1$  and  $H_3$  are burning or not. Each agent gets a noisy observation of one of these houses, which defines its type. In particular, agent 1 can observe flames ( $F_1$ ) or not ( $N_1$ ) at  $H_1$  and agent 2 can observe ( $F_2$ ) or not ( $N_2$ ) at  $H_3$ . The probability of making the correct observation is 0.9. Table 1 shows the resulting probabilities of joint types conditional on the state. The table also shows the *a priori* state distribution—it is most likely that none of the neighboring houses are on fire and  $H_3$  has a slightly higher probability of being on fire than  $H_1$ —and the resulting probability distribution over joint types, computed by marginalizing over states:  $\Pr(\theta) = \sum_s \Pr(\theta|s) \Pr(s)$ . Finally, each agent generates a +2 payoff for the team by fighting fire at a burning house. However, payoffs are sub-additive: if both agents fight fire at the same house (i.e., at  $H_2$ ), a payoff of +3 is generated. Fighting fire at a house that is not burning does not generate any payoff. Table 2 summarizes all the possible payoffs.

These rewards can be converted to the  $u(\theta, \mathbf{a})$  format by computing the conditional state probabilities  $\Pr(s|\theta)$  using Bayes' rule and taking the expectation over states:

$$u(\theta, \mathbf{a}) = \sum_s u(s, \mathbf{a}) \cdot \Pr(s|\theta). \quad (2.4)$$

The result is a fully specified Bayesian game whose payoff matrix is shown in Table 3.

$\theta_1$	$\theta_2$	$F_2$		$N_2$	
		$H_2$	$H_3$	$H_2$	$H_3$
$F_1$	$H_1$	3.414	2.032	3.14	1.22
	$H_2$	3	3.543	3	2.08
$N_1$	$H_1$	2.058	1.384	2.032	0.079
	$H_2$	3	3.326	3	2.047

Table 3: The Bayesian game payoff matrix for the two-agent firefighting problem.

### 2.3.2 Solution Concepts

In a BG, the concept of NE is replaced by a *Bayesian Nash equilibrium (BNE)*. A profile of policies  $\beta = \langle \beta_1, \dots, \beta_n \rangle$  is a BNE when no agent  $i$  has an incentive to switch its policy  $\beta_i$ , given the policies of the other agents  $\beta_{\neq i}$ . This occurs when, for each agent  $i$  and each of its types  $\theta_i$ ,  $\beta_i$  specifies the action that maximizes its expected value. When a Bayesian game is collaborative, the characterization of a BNE is simpler. Let the *value* of a joint policy be its expected payoff:

$$V(\beta) = \sum_{\theta \in \Theta} \Pr(\theta) u(\theta, \beta(\theta)), \quad (2.5)$$

where  $\beta(\theta) = \langle \beta_1(\theta_1), \dots, \beta_n(\theta_n) \rangle$  is the joint action specified by  $\beta$  for joint type  $\theta$ . Furthermore, let the *contribution* of a joint type be:

$$C_\theta(\mathbf{a}) \equiv \Pr(\theta) u(\theta, \mathbf{a}). \quad (2.6)$$

The value of a joint policy  $\beta$  can be interpreted as a sum of contributions, one for each joint type. The BNE of a CBG maximizes a sum of such contributions.

**Theorem 2.1.** *The Bayesian Nash equilibrium of a CBG is:*

$$\beta^* = \arg \max_{\beta} V(\beta) = \arg \max_{\beta} \sum_{\theta \in \Theta} C_\theta(\beta(\theta)), \quad (2.7)$$

*which is a Pareto-optimal (Bayesian) Nash equilibrium (PONE).*

*Proof.* A CBG  $G$  can be reduced to a CSG  $G'$  where each action of  $G'$  corresponds to a policy of  $G$ . Furthermore, in  $G'$ , a joint action  $\mathbf{a}'$  corresponds to a joint policy of  $G$  and the payoff of a joint action  $u'(\mathbf{a}')$  corresponds to the value of the joint policy. As explained in Sec. 2.1.2, a PONE for a CSG is a maximizing entry, which corresponds to (2.7). For a more formal proof, see (Oliehoek et al., 2008b).  $\square$

### 2.3.3 Solving CBGs

Although the characterization of a PONE is simple, finding one is intractable in general. In fact, a CBG is equivalent to a *team decision problem*, which is NP-hard (Tsitsiklis and Athans, 1985).

Since a CBG is an instance of a (non-collaborative) BG, solution methods for the latter apply. A common approach is to convert a BG  $G$  to an SG  $G'$ , as in the proof of Theorem 2.1. An action  $a'_i$  in  $G'$  correspond to a policy  $\beta_i$  in  $G$ ,  $a'_i \equiv \beta_i$ , and the payoff of a joint action in  $G'$  equals the expected payoff of the corresponding joint BG policy  $u'(\mathbf{a}') \equiv V(\beta)$ . However, since the number of policies for an agent in a BG is exponential in the number of types, the conversion to an SG leads to an exponential blowup in size. When applying this procedure in the collaborative case (i.e., to a CBG), the result is a CSG to which the trivial algorithm applies. In

effect, since joint actions correspond to joint BG-policies, this procedure corresponds to brute-force evaluation of all joint BG-policies.

A different approach to solving CBGs is *alternating maximization (AM)*. Starting with a random joint policy, each agent iteratively computes a best response policy for each of its types. In this way the agents hill-climb towards a local optimum. While the method guarantees finding an NE, it can not guarantee finding a PONE and there is no bound on the quality of the approximation. By starting from a specially constructed starting point, it is possible to give some guarantees on the quality of approximation (Cogill and Lall, 2006). These guarantees, however, degrade exponentially as the number of agents increases.

Finally, recent work shows that the additive structure of the value function (2.7) can be exploited by heuristic search to greatly speed up the computation of optimal solutions (Oliehoek et al., 2010). Furthermore, the point-based backup operation in a Dec-POMDP—which can be interpreted as a special case of CBG—can be solved using state-of-the-art weighted constraint satisfaction methods (Kumar and Zilberstein, 2010), also providing significant increases in performance.

### 3 Exploiting Independence in Collaborative Bayesian Games

The primary goal of this work is to find ways to efficiently solve large CBGs, i.e., CBGs with many agents, actions and types. None of the models presented in the previous section are adequate for the task. CGSGs, by representing independence between agents, allow solution methods that scale to many agents, but they do not model private information. In contrast, CBGs model private information but do not represent independence between agents. Consequently, CBG solution methods scale poorly with respect to the number of agents.

In this section, we propose a new model to address these issues. In particular, we make three main contributions. First, Sec. 3.1 distinguishes between two types of independence that can occur in CBGs: in addition to the *agent independence* that occurs in CGSGs, all CBGs possess *type independence*, an inherent consequence of imperfect information. Second, Sec. 3.2 proposes *collaborative graphical Bayesian games*, a new framework that models both these types of independence. Third, Sec. 3.4 describes solution methods for this model that use a novel factor graph representation to capture both agent and type independence such that they can be exploited by NDP and MAX-PLUS. We prove that, while the computational cost of NDP applied to such a factor graph remains exponential in the number of individual types, MAX-PLUS is tractable for small local neighborhoods.

#### 3.1 Agent and Type Independence

As explained in Sec. 2.2, in many CSGs, agent interactions are sparse. The resulting independence, which we call *agent independence*, has long been exploited to compactly represent and more efficiently solve games with many agents, as in the CGSG model.

While many CBGs also possess agent independence, the CBG framework provides no way to model or exploit it. In addition, regardless of whether they have agent independence, *all* CBGs possess a second kind of independence, which we call *type independence*, that is an inherent consequence of imperfect information. Unlike agent independence, type independence is captured in the CBG model and can thus be exploited.

Type independence, which we originally identified in (Oliehoek, 2010), is a result of the additive structure of a joint policy’s value (shown in (2.7)). The key insight is that each of the contribution terms from (2.6) depends only on the action selected for some individual types. In particular, the action  $\beta_i(\theta_i)$  selected for type  $\theta_i$  of agent  $i$  affects only the contribution terms whose joint types involve  $\theta_i$ .

For instance, in the two-agent firefighting problem, one possible joint type is  $\theta = \langle N, N \rangle$  (neither agent observes flames). Clearly, the action  $\beta_1(F)$  that agent 1 selects when it has type  $F$  (it observes flames), has no effect on the contribution of this joint type.

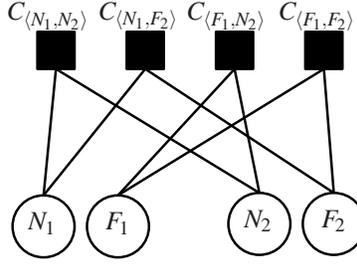


Figure 3: A factor graph of the two-agent firefighting problem, illustrating the type independence inherent in CBGs. The action chosen for an individual type  $\theta_i$  affects only a subset of contribution factors. For instance, the action that agent 1 selects when it has type  $N$  affects only the contribution factors  $C_{\langle N_1, N_2 \rangle}$  and  $C_{\langle N_1, F_2 \rangle}$  in which it has that type.

As illustrated in Fig. 3, this type of structure can also be represented by a factor graph with one set of nodes for all the contributions (corresponding to joint types) and another set for all the individual types of all the agents. Unlike the representation that results from reducing a BG to an SG played by *agent-type* combinations (Osborne and Rubinstein, 1994), this factor graph does not completely ‘flatten’ the utility function. On the contrary, it explicitly represents the contributions of each joint type, thereby capturing type independence. The distinction between agent and type independence is summarized in the following observation.

**Observation 1.** CBGs can possess two different types of independence:

1. Agent independence: the payoff function is additively decomposed over local payoff functions, each specified over only a subset of agents, as in CGSGs.
2. Type independence: only one type per agent is actually realized, leading to a value function that is additively decomposed over contributions, each specified over only a subset of types.

The consequence of this distinction is that neither the CGSG nor CBG model is adequate to model complex games with imperfect information. To scale to many agents, we need a new model that expresses (and therefore makes it possible to exploit) both types of independence. In the rest of this section, we propose a model that does this and show how both types of independence can be represented in a factor graph which, in turn, can be solved using NDP and MAX-PLUS.

### 3.2 The Collaborative Graphical Bayesian Game Model

A collaborative graphical Bayesian game is a CBG whose common payoff function decomposes over a number of local payoff functions (as in a CGSG).

**Definition 3.1.** A *collaborative graphical Bayesian game (CGBG)* is a tuple  $\langle \mathcal{D}, \mathcal{A}, \Theta, \mathcal{P}, \mathcal{U} \rangle$ , with:

- $\mathcal{D}, \mathcal{A}, \Theta$  as in a Bayesian game,
- $\mathcal{P} = \{\Pr(\Theta_1), \dots, \Pr(\Theta_p)\}$  is a set of consistent *local probability distributions*,
- $\mathcal{U} = \{u^1, \dots, u^p\}$  is the set of  $p$  *local payoff functions*. These correspond to a set  $\mathcal{E}$  of hyper-edges of an interaction graph, such that the total team payoff can (with some abuse of notation) be written as  $u(\theta, \mathbf{a}) = \sum_{e \in \mathcal{E}} u^e(\theta_e, \mathbf{a}_e)$ .

A CGBG is collaborative because all agents share the common payoff function  $u(\theta, \mathbf{a})$ . It is also graphical because this payoff function decomposes into a sum of local payoff functions, each of which depends on only a subset of agents.<sup>5</sup> As in CGSGs, each local payoff function  $u^e$  has scope  $\mathbb{A}(u^e)$ , which can be expressed in an interaction hyper-graph  $IG = \langle \mathcal{D}, \mathcal{E} \rangle$  with one hyper-edge for each  $e \in \mathcal{E}$ . Strictly speaking, an edge corresponds to the scope of a local payoff function, i.e., the set of agents that participate in it (as in ‘ $\mathbf{a}_e$ ’), but we will also use  $e$  to index the sets of hyper-edges and payoff functions (as in  $u^e$ ).

Each local payoff function depends not only on the local joint action  $\mathbf{a}_e$ , but also on the *local joint type*  $\theta_e$ , i.e., the types of the agents in  $e$  (i.e., in  $\mathbb{A}(u^e)$ ). Furthermore, each local probability function  $\Pr(\theta_e)$  specifies the probability of each local joint type. The goal is to maximize the expected sum of rewards:

$$\beta^* = \arg \max_{\beta} \sum_{\theta} \Pr(\theta) u(\theta, \beta(\theta)) = \arg \max_{\beta} \sum_{e \in \mathcal{E}} \sum_{\theta_e} \Pr(\theta_e) u^e(\theta_e, \beta_e(\theta_e)) \quad (3.1)$$

where  $\beta_e(\theta_e)$  is the local joint action under policy  $\beta$  given local joint type  $\theta_e$ .

In principle, the local probability functions can be computed from the full joint probability function  $\Pr(\Theta)$ . However, doing so is generally intractable as it requires marginalizing over the types that are not in scope. By including  $\mathcal{P}$  in the model, we implicitly assume that  $\Pr(\Theta)$  has a compact representation that allows for efficient computation of  $\Pr(\theta_e)$ , e.g., by means of Bayesian networks (Pearl, 1988; Bishop, 2006) or other graphical models.

Not all probability distributions over joint types will admit such a compact representation. However, those that do not will have a size exponential in the number of agents and thus cannot even be represented, much less solved, efficiently. Thus, the assumption that these local probability functions exist is minimal in the sense that it is a necessary condition for solving the game efficiently. Note, however, that it is not a sufficient condition. On the contrary, the computational advantage of the methods proposed below results from the agent and type independence captured in the resulting factor graph, not the existence of local probability functions.

### Example: Generalized Fire Fighting

As an example, consider GENERALIZED FIRE FIGHTING, which is like the two-agent firefighting problem of Sec. 2.3.1 but with  $n$  agents. In this version there are  $N_H$  houses and the agents are physically distributed over the area. Each agent gets an observation of the  $N_O$  nearest houses and may choose to fight fire at any of the  $N_A$  nearest houses. For each house  $H$ , there is a local payoff function involving the agents in its neighborhood (i.e., of the agents that can choose to fight fire at  $H$ ). These payoff functions yield sub-additive rewards similar to those in Table 2. The type of each agent  $i$  is defined by the  $N_O$  observations it receives from the surrounding houses:  $\theta_i \in \{F_i, N_i\}^{N_O}$ . The probability of each type depends on the probability that the surrounding houses are burning. As long as those probabilities can be compactly represented, the probabilities over types can be too. Fig. 4 illustrates the case where  $N_H = 4$  and  $n = 3$ . Each agent can go to the  $N_A = 2$  closest houses. In this problem, there are 4 local payoff functions, one for each house, each with limited scope. Note that the payoff functions for the first and the last house include only one agent, which means their scopes are proper subsets of the scopes of other payoff functions (those for houses 2 and 3 respectively). Therefore, they can be included in those functions, reducing the number of local payoff functions in this example to two: one in which agents 1 and 2 participate, and one in which agents 2 and 3 participate.

### 3.3 Relationship to Other Models

To provide a better understanding of the CGBG model, we elaborate on its relationship with existing models. Just as CGSGs are related to graphical games, CGBGs are related to graphical BGs (Soni et al., 2007). However,

<sup>5</sup>Arguably, since all CGBGs have type independence, they are in some sense already graphical, as illustrated in Fig. 3. However, to be consistent with the literature, we use the term graphical here to indicate agent independence.

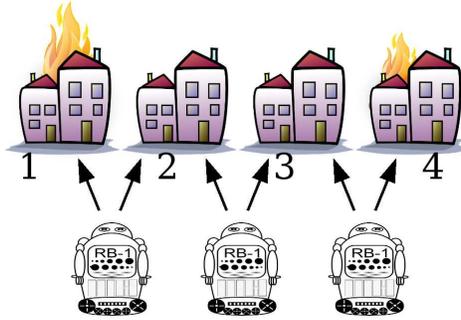


Figure 4: Illustration of GENERALIZED FIRE FIGHTING with  $N_H = 4$  and  $n = 3$ .

as before, there is a crucial difference in the meaning of the term ‘graphical’. In CGBGs, all agents participate in a common payoff function that decomposes into local payoff functions, each involving subsets of agents. In contrast, graphical BGs are not necessarily collaborative and the individual payoff functions involve subsets of agents. Since these individual payoff functions do not decompose, CGBGs are not a special case of GBGs but rather a unique, novel formalism. In addition, the graphical BGs considered by Soni et al. (2007) make much more restrictive assumptions on the type probabilities, allowing only *independent type distributions* (i.e.,  $\Pr(\Theta)$  is defined as the product of individual type probabilities  $\Pr(\Theta_i)$ ) and assuming *conditional utility independence* (i.e., the payoff of an agent  $i$  depends only on its own type, not that of other agents:  $u_i(\theta_i, \mathbf{a})$ ).

More closely related is the multiagent influence diagram (MAID) framework that extends decision diagrams to multiagent settings (Koller and Milch, 2003). In particular, a MAID represents a decision problem with a Bayesian network that contains a set of chance nodes and, for each agent, a set of decision and utility nodes. As in a CGBG, the individual payoff function for each agent is defined as the sum of local payoffs (one for each utility node of that agent). On the one hand, MAIDs are more general than CGBGs because they can represent non-identical payoff settings (though it would be straightforward to extend CGBGs to such problems). On the other hand, CGBGs are more general than MAIDs since they allow any representation of the distribution over joint types (e.g., a Markov random field), as long as the local probability distributions can be computed efficiently.

When the local probabilities  $\mathcal{P}$  are explicitly computed, a CGBG can be represented as a MAID, as illustrated in Fig. 5. In this MAID, both utility nodes are associated with all agents, such that each agent’s goal is to optimize the sum  $u^1 + u^2$  and the MAID is collaborative. The probability of an individual type is a deterministic function of all the incoming local joint types, e.g., the probability of a particular value of  $\theta_2$  is 1 if (and only if) both  $\theta_{12}$  and  $\theta_{23}$  specify that value.

However, the resulting MAID’s *relevance graph* (Koller and Milch, 2003), which indicates which decisions influence each other, consists of a single strongly connected component. Consequently, the divide and conquer solution method proposed by Koller and Milch offers no speedup over brute-force evaluation of all the joint policies. In the following section, we propose methods to overcome this problem and solve CGBGs efficiently.

### 3.4 Solution Methods

Solving a CGBG amounts to finding the maximizing joint policy as expressed by (3.1). As mentioned in Sec. 2.3.3, it is possible to convert a BG to an SG in which the actions correspond to BG policies. In previous work (Oliehoek et al., 2008c), we applied similar transformations to CGBGs, yielding CGSGs to which all the solution methods mentioned in Sec. 2.2 are applicable. Alternatively, it is possible to convert to a

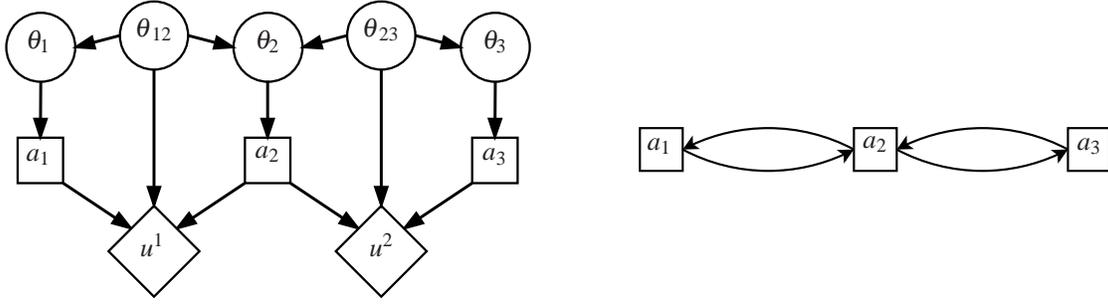


Figure 5: At left, GENERALIZED FIRE FIGHTING represented as a MAID. At right, the resulting relevance graph, which has only one strongly connected component containing all decision variables.

(non-collaborative) graphical BG (Soni et al., 2007) and apply the proposed solution method. Under the hood, however, this method converts to a graphical SG.

The primary limitation of all of the options mentioned above is that they exploit only agent independence, not type independence. In fact, converting a CGBG to a CGSG has the effect of stripping all type independence from the model. To see why, note that type independence in a CBG occurs as a result of the form of the payoff function  $u(\theta, \beta(\theta))$ . In other words, the payoff depends on the joint action, which in turn depends only on the joint type that occurs. Converting to a CSG produces a payoff function that depends on the joint action selected for *all possible* joint types, effectively ignoring type independence. A direct result is that the solution methods have an exponential dependence on the number of types.

In this section, we propose a new approach to solving CGBGs that avoids this problem by exploiting both kinds of independence. The main idea is to represent the CGBG using a novel factor graph formulation that neatly captures both agent and type independence. The resulting factor graph can then be solved using methods such as NDP and MAX-PLUS.

To enable this factor graph formulation, we define a *local contribution* as follows:

$$C_{\theta_e}^e(\mathbf{a}_e) \equiv \Pr(\theta_e)u^e(\theta_e, \mathbf{a}_e) \quad (3.2)$$

Using this notation, the solution of the CGBG is

$$\beta^* = \arg \max_{\beta} \sum_{e \in \mathcal{E}} \sum_{\theta_e} C_{\theta_e}^e(\beta_e(\theta_e)). \quad (3.3)$$

Thus, the solution corresponds to the maximum of an additively decomposed function containing a contribution for each local joint type  $\theta_e$ . This can be expressed in a factor graph with one set of nodes for all the contributions and another for all the individual types of all the agents. An individual type  $\theta_i$  of an agent  $i$  is connected to a contribution  $C_{\theta_e}^e$  only if  $i$  participates in  $u^e$  and  $\theta_e = \langle \theta_j \rangle_{j \in \mathbb{A}(u^e)}$  specifies  $\theta_i$  for agent  $i$ , as illustrated in Fig. 6. We refer to this graph as the *agent and type independence (ATI) factor graph*.<sup>6</sup> Contributions are separated, not only by the joint type to which they apply, but also by the local payoff function to which they contribute. Consequently, both agent and type independence are naturally expressed. In the next two subsections, we discuss the application of NDP and MAX-PLUS to this factor graph formulation in order to efficiently solve CGBGs.

<sup>6</sup>In previous work, we referred to this as the ‘type-action’ factor graph, since its variables correspond to actions selected for individual types (Oliehoek, 2010).

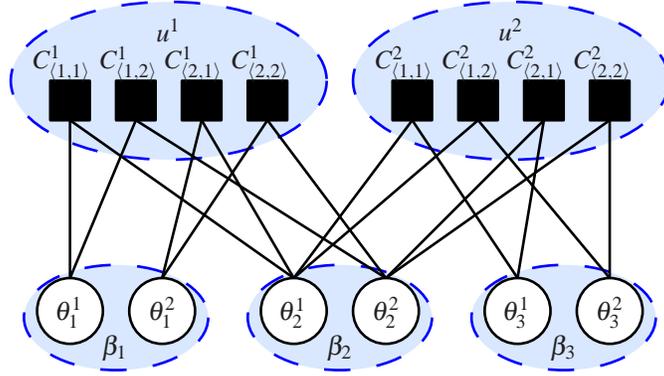


Figure 6: Factor graph for GENERALIZED FIRE FIGHTING with three agents, two types per agent, and two local payoff functions. Agents 1 and 2 participate in payoff function  $u^1$  (corresponding to the first four contributions), while agents 2 and 3 participate in  $u^2$  (corresponding to the last four contributions). The factor graph expresses both agent independence (e.g., agent 1 does not participate in  $u^2$ ) and type independence (e.g, the action agent 1 selects when it receives observation  $\theta_1^1$  affects only the first 2 contributions).

### 3.4.1 Non-Serial Dynamic Programming for CGBGs

Non-serial dynamic programming (NDP) (Bertele and Brioschi, 1972) can be used to find the maximum configuration of a factor graph. In the *forward pass*, the variables in the factor graph are eliminated one by one according to some prespecified order. Eliminating the  $k$ th variable  $v$  involves collecting all the factors in which it participates and replacing them with a new factor  $f^k$  that represents the sum of the removed factors, given that  $v$  selects a best response. Once all variables are eliminated, the *backwards pass* begins, iterating through the variables in reverse order of elimination. Each variable selects a best response to the variables already visited, eventually yielding an optimal joint policy.

The maximum number of agents participating in a factor encountered during NDP is known as the induced width  $w$  of the ordering. The following result is well-known (see for instance (Dechter, 1999)):

**Theorem 3.1.** *NDP requires exponential time and space in the induced width  $w$ .*

Even though NDP is still exponential, for sparse problems the induced width is much smaller than the total number of variables  $V$ , i.e.,  $w \ll V$ , leading to an exponential speed up over naive enumeration over joint variables.

In previous work (Oliehoek et al., 2008c), we used NDP to optimally solve CGBGs. However, NDP was applied to the *agent independence (AI)* factor graph (e.g., as in Fig. 2b) that results from converting the CGBG to a CGSG. Consequently, only agent independence was exploited. In principle, we should be able to improve performance by applying NDP to the ATI factor graph introduced above, thereby exploiting both agent and type independence. Fig. 7 illustrates a few steps of the resulting algorithm.

However, there are two important limitations of the NDP approach. First, the computational complexity is exponential in the induced width, which in turn depends on the order in which the variables are eliminated. Determining the optimal order (which Bertelè and Brioschi (1973) call the *secondary optimization problem*) is NP-complete (Arnborg et al., 1987). While there are heuristics for determining the order, NDP scales poorly in practice on densely connected graphs (Kok and Vlassis, 2006). Second, because of the particular shape that type independence induces on the factor graph, we can establish the following:

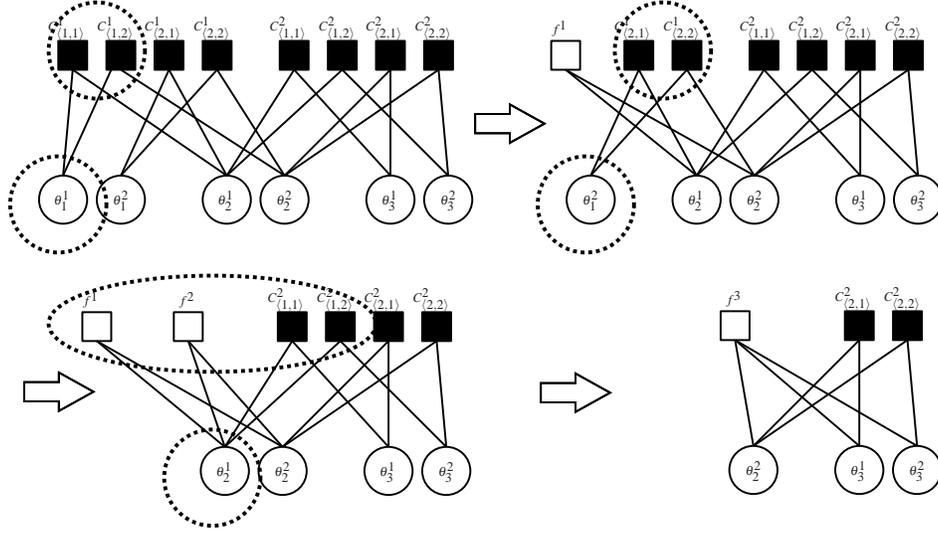


Figure 7: A few steps of NDP run on the factor graph of Fig. 6. Variables are eliminated from left to right. Dotted ellipses indicate the part of the factor graph to be eliminated.

**Theorem 3.2.** *The induced width of an ATI factor graph is exponential in the number of individual types:  $w \propto O(\exp(|\Theta_*|))$ , where  $\Theta_*$  denotes the largest individual type set.*

*Proof.* Let us consider the first elimination step of NDP for an arbitrary  $\theta_i^m$  (i.e., an arbitrary variable). Now, each edge  $e \in \mathcal{E}$  in which it participates induces  $O(|\Theta_*|^{|e|-1})$  contributions to which it is connected: one contribution for each profile  $\theta_{e \setminus i}$  of types of the other agents in  $e$ . As a result, the new factor  $f^1$  is connected to all types of the neighbors in the interaction hyper-graph. The number of such types is at least  $\Theta_*$ .  $\square$

Note that  $|\Theta_*|$  is not the only term that determines  $w$ ; the number of edges  $e \in \mathcal{E}$  in which agents participate as well as the elimination order still matter. In particular, let  $k$  denote the maximum degree of a contribution factor, i.e., the largest local scope  $k = \max_{e \in \mathcal{E}} |\mathbb{A}(u^e)|$ . Clearly, since there is a factor that has degree  $k$ , we have that  $w \geq k$ .

**Corollary 1.** *The computational complexity of NDP applied to an ATI factor graph is exponential in the number of individual types.*

*Proof.* This follows directly from theorems 3.1 and 3.2.  $\square$

Therefore, even given the ATI factor graph formulation, it seems unlikely that NDP will prove useful in exploiting type independence. In particular, we hypothesize that NDP applied to the ATI factor graph will not perform significantly better than NDP on the AI factor graph. In fact, it is possible that the former performs worse than the latter. This is illustrated in the last elimination step shown in Fig. 7. A factor  $f^3$  is introduced with degree  $w = 3$  and size  $|\mathcal{A}_*|^w$ . In contrast, performing NDP on the AI factor graph using the ‘same’ left-to-right ordering has induced width  $w = 1$  and the size of the factors constructed is  $(|\mathcal{A}_*|^2)^w$  (where  $2 = |\Theta_*|$ ).

### 3.4.2 Max-Plus for CGBGs

In order to more effectively exploit type independence, we consider a second approach in which the factor graph is solved using the MAX-PLUS message passing algorithm (Pearl, 1988; Wainwright et al., 2004; Kok and Vlassis, 2005; Vlassis, 2007). MAX-PLUS was originally proposed by Pearl (1988) under the name *belief revision* to compute the maximum *a posteriori* probability configurations in Bayesian networks. The algorithm is also known as max-product or min-sum (Wainwright et al., 2004) and is a special case of the *sum-product* algorithm (Kschischang et al., 2001)—also referred to as *belief propagation* in probabilistic domains. MAX-PLUS can be implemented in either a centralized or decentralized way (Kok and Vlassis, 2006). However, since we assume planning takes place in a centralized off-line phase, we consider only the former.

MAX-PLUS algorithm is an appealing choice for several reasons. First, on structured problems it has been shown to achieve excellent performance in practice (Kschischang et al., 2001; Kok and Vlassis, 2006; Kuyer et al., 2008). Second, unlike NDP, it is an *anytime* algorithm that can provide results after each iteration of the algorithm, not only at the end (Kok and Vlassis, 2006). Third, as we show below, its computational complexity is exponential only in the size of the largest local payoff function’s scope, which is fixed for a given CGBG.

At an intuitive level, MAX-PLUS works by iteratively sending messages between the factors, corresponding to contributions, and variables, corresponding to (choices of actions for) types. These messages encode how much payoff the sender expects to be able to contribute to the total payoff. In particular, a message sent from a type  $i$  to a contribution  $j$  encodes, for each possible action, the payoff it expects to contribute. This is computed as the sum of the incoming messages from other contributions  $k \neq j$ . Similarly, a message sent from a contribution to a type  $i$  encodes the payoff it can contribute conditioned on each available action to the agent with type  $i$ .<sup>7</sup>

MAX-PLUS iteratively passes these messages over the edges of the factor graph. Within each iteration, the messages are sent either in parallel or sequentially with a fixed or random ordering. When run on an acyclic factor graph (i.e., a tree), it is guaranteed to converge to an optimal fixed point (Pearl, 1988; Wainwright et al., 2004). In cyclic factor graphs, such as those defined in Sec. 3.4, there are no guarantees that MAX-PLUS will converge.<sup>8</sup> However, experimental results have demonstrated that it works well in practice even when cycles are present (Kschischang et al., 2001; Kok and Vlassis, 2006; Kuyer et al., 2008). This requires normalizing the messages to prevent them from growing ever larger, e.g. by taking a weighted sum of the new and old messages (damping).

As mentioned above, the computational complexity of MAX-PLUS on a CGBG is exponential only in the size of the largest local payoff function’s scope. More precisely, we show here that this claim holds for one iteration of MAX-PLUS. In general, it is not possible to bound the number of iterations, since MAX-PLUS is not guaranteed to converge. However, by applying renormalization and/or damping, MAX-PLUS converges quickly in practice. Also, since MAX-PLUS is an anytime algorithm, it is possible to limit the number of iterations to a constant number.

**Theorem 3.3.** *One iteration of MAX-PLUS run on the factor graph constructed for a CGBG is tractable for small local neighborhoods, i.e., the only exponential dependence is in the size of the largest local scope.*

*Proof.* Lemma 8.1 in the appendix characterizes the complexity of one iteration of MAX-PLUS as

$$O\left(m^k \cdot k^2 \cdot l \cdot F\right), \tag{3.4}$$

where, for a factor graph for a CGBG, the interpretation of the symbols is as follows:

<sup>7</sup> For a detailed description of how the messages are computed, see (Oliehoek, 2010, Sec. 5.5.3).

<sup>8</sup> However, recent variants of the message passing approach have slight modifications that yield convergence guarantees (Globerson and Jaakkola, 2008). Since we found that regular MAX-PLUS performs well in our experimental setting, we do not consider such variants in this article.

- $m$  is the maximum number of values a type variable can take. It is given by  $m = |\mathcal{A}_*|$ , the size of the largest action set.
- $k$  is the maximum degree of a contribution (factor), given by the largest local scope  $k = \max_{e \in \mathcal{E}} |\mathbb{A}(u^e)|$ .
- $l$  is the maximum degree of a type (variable). Again, each local payoff function  $e \in \mathcal{E}$  in which it participates induces  $O(|\Theta_*|^{k-1})$  contributions to which it is connected. Let  $\rho_*$  denote the maximum number of edges in which an agent participates. Then  $l = O(\rho_* \cdot |\Theta_*|^{k-1})$ .
- $F = O(\rho \cdot |\Theta_*|^k)$  is the number of contributions, one for each *local* joint type.

By substituting these numbers and reordering terms we get that one iteration of MAX-PLUS for a CGBG has cost:

$$O\left(|\mathcal{A}_*|^k \cdot k^2 \cdot \rho \rho_* |\Theta_*|^{2k-1}\right), \quad (3.5)$$

Thus, in the worst case, one iteration of MAX-PLUS scales polynomially with respect to the number of local payoff functions  $\rho$  and the largest sets of actions  $|\mathcal{A}_*|$  and types  $|\Theta_*|$ . It scales exponentially in  $k$ .  $\square$

Given this result, we expect that MAX-PLUS will prove more effective than NDP at exploiting type independence. In particular, we hypothesize that MAX-PLUS will perform better when applied to the ATI factor graph instead of the AI factor graph and that it will perform better than NDP applied to the ATI factor graph. In the following section, we present experiments evaluating these hypotheses.

### 3.5 Random CGBG Experiments

To assess the relative performance of NDP and MAX-PLUS, we conduct a set of empirical evaluations on randomly generated CGBGs. We use randomly generated games because they allow for testing on a range of different problem parameters. In particular, we are interested in the effect of scaling the number of agents  $n$ , the number of types  $|\Theta_i|$  that each agent has, the number of actions for each agent  $|\mathcal{A}_i|$ , as well as the number of agents involved in each payoff function,  $|\mathbb{A}(e)|$ . We assume each payoff function has an equal number of agents and refer to this property as  $k = \max_{e \in \mathcal{E}} |\mathbb{A}(e)|$ , as in Theorem 3.3.

Furthermore, we empirically evaluate the influence of exploiting both agent and type independence versus exploiting only one or the other. We do so by running both NDP and MAX-PLUS on the agent-independence (AI) factor graph (Fig. 2b), the type-independence (TI) factor graph (Fig. 3), and the agent and type independence (ATI) factor graph (Fig. 6).

These experiments serve three main purposes. First, they empirically validate Theorems 3.2 and 3.3, confirming the difference in computational complexity between NDP and MAX-PLUS. Second, they quantify the magnitude of the difference in runtime performance between these two methods. Third, they shed light on the quality of the solutions found by MAX-PLUS, which is guaranteed to be optimal only on tree-structured graphs.

#### 3.5.1 Experimental Setup

For simplicity, when generating CGBGs, we assume that 1) the scopes of the local payoff functions have the same size  $k$ , 2) the individual action sets have the same size  $|\mathcal{A}_i|$ , and 3) the individual type sets have the same size  $|\Theta_i|$ . For each set of parameters, we generate 1,000 CGBGs on which to test.

Each game is generated following a procedure similar to that used by Kok and Vlassis (2006) for generating CGSGs.<sup>9</sup> We start with a set of  $n$  agents with no local payoff functions defined, i.e., they form an interaction

<sup>9</sup>The main difference is in the termination condition: we stop adding edges when the interaction graph is fully connected instead of adding a pre-defined number of edges.

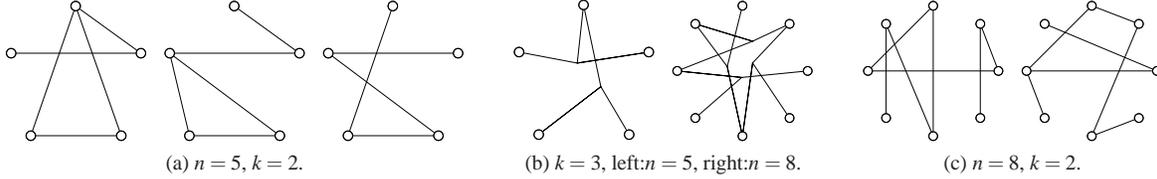


Figure 8: Example interaction hypergraphs of randomly generated CGBGs for different parameter settings.

hypergraph with no edges. As long as the interaction hypergraph is not yet connected, i.e., there does not exist a path between every pair of agents, we add a local payoff function involving  $k$  agents.

As a result, the number of edges in different CGBGs generated for the same parameter setting may differ significantly. The  $k$  agents that participate in a new edge are selected uniformly at random from the subset of agents involved in the fewest number of edges. Payoffs  $u^e(\theta_e, \mathbf{a}_e)$  are drawn from a normal distribution  $\mathcal{N}(0,1)$ , and the local joint type probabilities  $\Pr(\theta_e)$  are drawn from a uniform distribution and then normalized. This algorithm results in fully connected interaction hypergraphs that are balanced in terms of the number of payoff functions in which each agent participates. Fig. 8 shows some examples of the interaction hypergraphs generated.

We test the following methods:

**NDP** Non-serial dynamic programming (see Sec. 3.4.1), run on the agent-independence factor graph (NDP-AI), the type-independence factor graph (NDP-TI), and the proposed agent-type independence factor graph (NDP-ATI).

**MP** MAX-PLUS with the following parameters: 10 restarts, 25 maximum iterations, sequential random message passing scheme, damping factor 0.2. Analogous to NDP, there are three variations MAX-PLUS-AI, MAX-PLUS-TI, and MAX-PLUS-ATI.

**BAGABAB** Bayesian game branch and bound (BAGABAB) is a fast method for optimally solving CGBs (Oliehoek et al., 2010). It performs heuristic search over partially specified policies.<sup>10</sup>

**ALTMAX** Alternating Maximization with 10 restarts (see Sec. 2.3.3).

**CE** Cross Entropy optimization (de Boer et al., 2005) is a randomized optimization method that maintains a distribution over joint policies. It works by iterating the following steps: 1) sampling a set of joint policies from the distribution 2) using the best fraction of samples to update the distribution. We used two parameter settings: one that gave good results according to (Oliehoek et al., 2008a) (CENORMAL), and one that is faster (CEFAST).<sup>11</sup>

All methods were implemented using the MADP Toolbox (Spaan and Oliehoek, 2008); the NDP and MAX-PLUS implementations also use LIBDAI (Mooij, 2008b). Experiments in this section are run on an Intel Core i5 CPU (2.67GHz) using Linux, and the timing results are CPU times. Each process is limited to 1GB of memory use and the computation time for solving a single CGBG is limited to 5s.

<sup>10</sup>In the experiments we used the ‘MaxContributionDifference’ joint type ordering and use the ‘consistent complete information’ heuristic.

<sup>11</sup>Both variants perform 10 restarts, use a learning rate of 0.2 and perform what (Oliehoek et al., 2008a) refer to as ‘approximate evaluation’ of joint policies. CENORMAL performs 300 and CEFAST 100 simulations per joint policy. CENORMAL performs  $I = 50$  iterations, in each of which  $N = 100$  joint policies are sampled of which  $N_b = 5$  policies are used to update the maintained distribution. CEFAST uses  $I = 15$ ,  $N = 40$ ,  $N_b = 2$ .

FG type	num. factors ( $F$ )	fact. size	fact. deg. ( $k$ )	num. vars	var. size ( $m$ )	var. deg. ( $l$ )
AI	$\rho$	$ \mathcal{A}_* ^{ \Theta_* k}$	$ e_* $	$n$	$ \mathcal{A}_* ^{ \Theta_* }$	$\rho_*$
TI	$ \Theta_* ^n$	$ \mathcal{A}_* ^n$	$n$	$n \Theta_* $	$ \mathcal{A}_* $	$ \Theta_* ^{n-1}$
ATI	$\rho \Theta_* ^k$	$ \mathcal{A}_* ^k$	$ e_* $	$n \Theta_* $	$ \mathcal{A}_* $	$ \Theta_* ^{k-1}$

Table 4: A characterization of the different types of factor graphs: agent-independence (AI), type-independence (TI), agent-type independence (ATI). The symbols relate to (3.4).

For each method, we report both the average payoff and the average CPU time needed to compute the solution. In the plots in this section, each data point represents an average over  $N_g = 1,000$  games. The reported payoffs are normalized with respect to those of MAX-PLUS-ATI. As such, the payoff of MAX-PLUS-ATI is always 1. Error bars indicate the standard deviation of the sampled mean  $\sigma_{\text{mean}} = \sigma / \sqrt{N_g}$ .

### 3.5.2 Comparing Methods

First, we compare NDP-ATI and MAX-PLUS-ATI with other methods that do not exploit a factor graph representation explicitly, the results of which are shown in Fig. 9. These results demonstrate that, as the number of agents increases, the average payoff of the approximate non-graphical methods goes down (Fig. 9a) while computation time goes up (Fig. 9b), given that the other parameters are fixed at  $k = 2$ ,  $|\Theta_i| = 3$ ,  $|\mathcal{A}_i| = 3$ . Note that a data point is not presented if the method exceeded the pre-defined resource limits on one or more test runs. For instance, BAGABAB can compute solutions only up to 4 agents. Also, Fig. 9b suggests that NDP-ATI would on average complete within the 5s deadline for 6 agents. However, because there is at least one run that does not, the data point is not included.

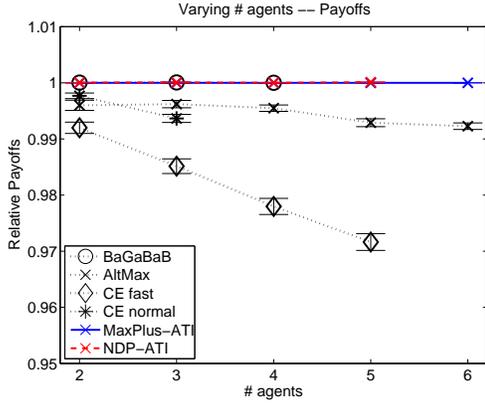
Next, we fix the number of agents to 5, and vary the number of actions  $|\mathcal{A}_i|$ . While CENORMAL never meets the time limit, the computation time that CEFASST requires is relatively independent of the number of actions (Fig. 9d). Payoff, however, drops sharply when the number of actions increases (Fig. 9c). The CE solvers maintain a fixed-size pool of possible solutions, which explains both phenomena: the same number of samples need to cover a larger search space, but the cost of evaluating each sample is relatively insensitive to  $|\mathcal{A}_i|$ .

Finally, we consider the behavior of the different methods when increasing the number of individual types  $|\Theta_i|$  (Fig. 9e and 9f). Since the number of policies for an agent is exponential in the number of types, the existing methods scale poorly. As established by Corollary 1, NDP’s computational costs also grow exponentially with the number of types. MAX-PLUS, in contrast, scales much better in the number of types. Looking at the quality of the found policies, we see that MAX-PLUS achieves the optimal value in these experiments, while the other approximate methods achieve lower values.

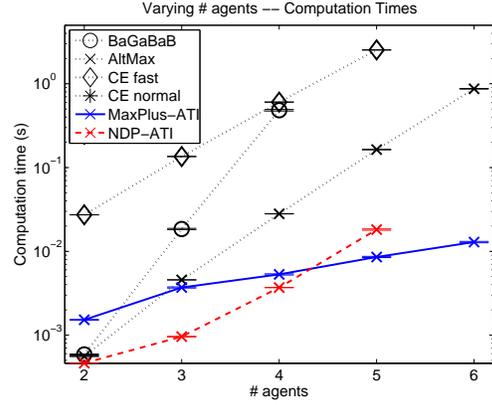
### 3.5.3 Comparing Factor-Graph Formulations

We now turn to a more in-depth analysis of the NDP and MAX-PLUS methods, in order to establish the effect of exploiting different types of independence. In particular, we test both methods on three different types of factor graphs: those with agent-independence (AI), type-independence (TI), and agent-type independence (ATI). Table 4 summarizes the types of factor graphs and the symbols used to describe their various characteristics.

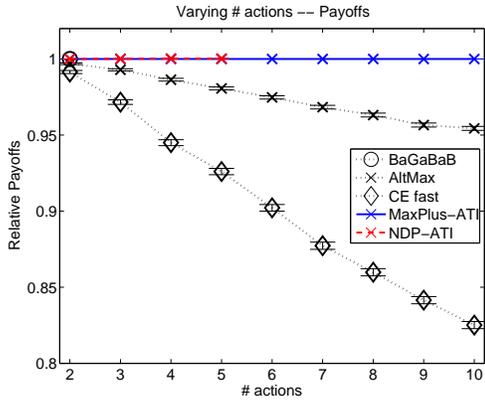
First, we consider scaling the number of agents, using the same parameters as in Fig. 9a and 9b. Fig. 10a shows the payoff of the different methods. The difference between them is not significant. However, Fig. 10b



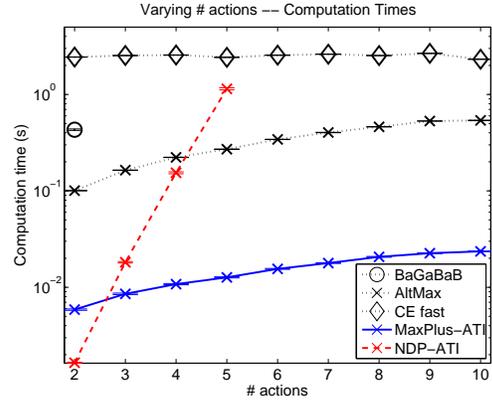
(a) Scaling  $n$ : Payoff  $s$  ( $k = 2, |\Theta_i| = 3, |\mathcal{A}_i| = 3$ ).



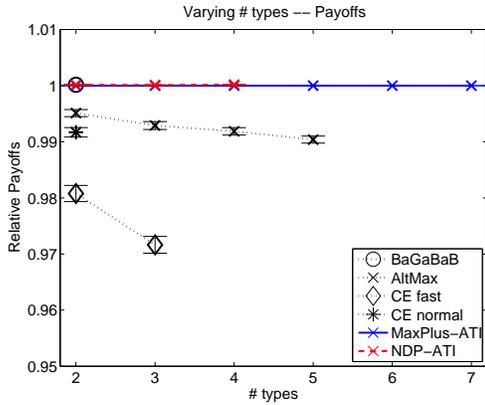
(b) Scaling  $n$ : Computation times ( $k = 2, |\Theta_i| = 3, |\mathcal{A}_i| = 3$ ).



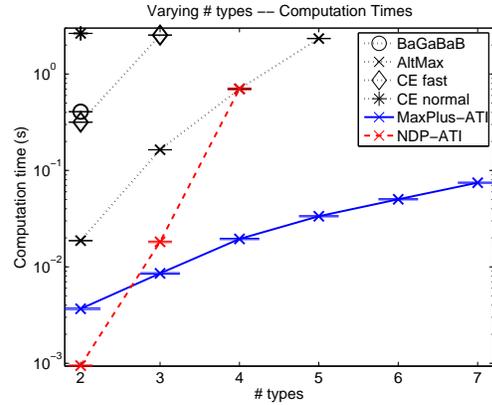
(c) Scaling  $|\mathcal{A}_i|$ : Payoff ( $k = 2, |\Theta_i| = 3, n = 5$ ).



(d) Scaling  $|\mathcal{A}_i|$ : Computation times ( $k = 2, |\Theta_i| = 3, n = 5$ ).



(e) Scaling  $|\Theta_i|$ : Payoff ( $k = 2, |\mathcal{A}_i| = 3, n = 5$ ).



(f) Scaling  $|\Theta_i|$ : Computation times ( $k = 2, |\mathcal{A}_i| = 3, n = 5$ ).

Figure 9: Comparison of MAX-PLUS-ATI and NDP-ATI with other methods, scaling the number of agents ((a) and (b)), the number of actions ((c) and (d)), and the number of types ((e) and (f)).

shows the computation times of the same methods. As expected, the methods that use only type independence scale poorly because the number of factors in their factor graph is exponential in the number of agents (Table 4).

Fig. 10c and 10d show similar comparisons for payoff functions involving three agents, i.e.,  $k = 3$  (example interaction hypergraphs are shown in Fig. 8b). The difference in payoff between NDP-ATI and MAX-PLUS-ATI is not significant (minimum p-value is 0.61907 for 6 agents). Differences with AM and the outlying points of MAX-PLUS-AI are significant (p-value  $< 0.05$ ). The NDP-AI and NDP-ATI methods scale to 6 agents, while MAX-PLUS-AI and MAX-PLUS-ATI scale beyond. The payoff of MAX-PLUS-AI is worse and more erratic than the payoff of MAX-PLUS-ATI. In this case, due to increased problem complexity, the MAX-PLUS methods typically do not attain the true optimum.

These experiments clearly demonstrate that only MAX-PLUS-AI and MAX-PLUS-ATI scale to larger numbers of agents. The poor scalability of the non-factor-graph methods is due to their failure to exploit the independence in CGBGs. The methods using TI factor graphs scale poorly because they ignore independence between agents. As hypothesized, NDP is not able to effectively exploit type independence and consequently NDP-ATI does not outperform NDP-AI. In fact, the experiments show that, in some cases, NDP-AI slightly outperforms NDP-ATI.

Fig. 10e and 10f show the performance of MAX-PLUS-AI and MAX-PLUS-ATI for games with  $k = 2$ ,  $|\Theta_i| = 4$ ,  $|a_i| = 4$  and larger numbers of agents, from  $n = 10$  up to  $n = 725$  (limited by the allocated memory space). For this experiment, the methods were allowed 30s per CGBG instance. Fig. 10e shows the absolute payoff obtained by both methods and the growth in the number of payoff functions (edges). The results demonstrate that the MAX-PLUS-ATI payoffs do not deteriorate when increasing the number of payoff functions. Instead, they increase steadily at a rate similar to the number of payoff functions. This is as expected, since more payoff functions means there is more reward to be collected. MAX-PLUS-AI scales only to 50 agents, and its payoffs are close to those obtained by MAX-PLUS-ATI. Fig. 10f provides clear experimental corroboration of Theorem 3.3 (which states that there is no exponential dependence on the number of agents), by showing scalability to 725 agents.

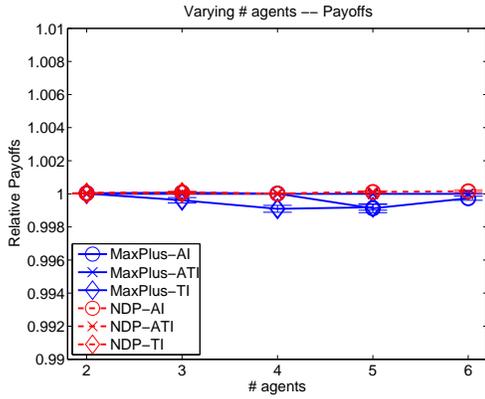
Analogously to Sec. 3.5.2, we compare the different factor-graph methods when increasing the number of actions and types. Fig. 11b shows that MAX-PLUS-ATI scales better in the number of actions while obtaining payoffs close to optimal (when available) and better than other MAX-PLUS variations (Fig. 11a) (differences are not significant). In fact, it is the only method whose computation time increases only slightly when increasing the number of actions: the size of each factor is only  $|\mathcal{A}_*|^k$  compared to  $|\mathcal{A}_*|^{|\Theta_*|^k}$  for AI and  $|\mathcal{A}_*|^n$  for TI (Table 4). In this case  $k = 2$  and  $n = 5$ , and in general  $k \ll n$  in the domains we consider.

When scaling the number of types (Fig. 11c and 11d), again there are no significant differences in payoffs. However, as expected given the lack of exponential dependence on  $|\Theta_i|$  (Table 4), MAX-PLUS-ATI performs much better in terms of computation times.

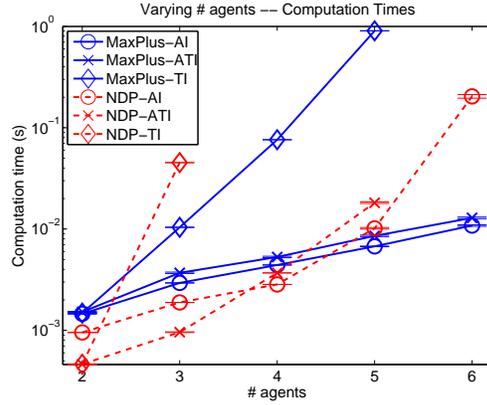
Overall, the results presented in this section demonstrate that the proposed methods substantially outperform existing solution methods for CBGs. In addition, the experiments confirm the hypothesis that NDP is not able to effectively exploit type independence, resulting in exponential scaling with respect to the number of types. MAX-PLUS on ATI factor graphs is able to effectively exploit both agent and type independence, resulting in much better scaling behavior with respect to all model parameters. Finally, the experiments showed that the value of the found solutions was not significantly lower than the optimal value and, in many case, significantly better than that found by other approximate solution methods.

### 3.6 Generalized Fire Fighting Experiments

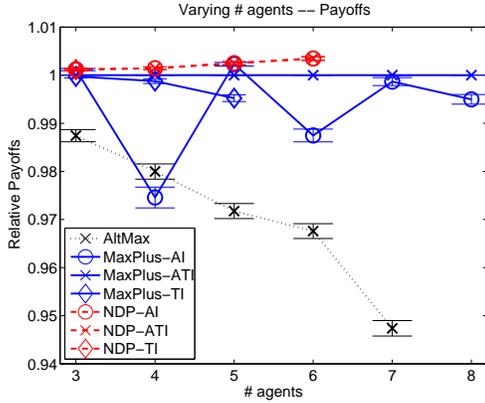
The results presented above demonstrate that MAX-PLUS-ATI can improve performance on a wide range of CGBGs. However, all of the CGBGs used in those experiments were randomly generated. In this section, we aim to demonstrate that the advantages of MAX-PLUS-ATI extend to a more realistic problem. To this end,



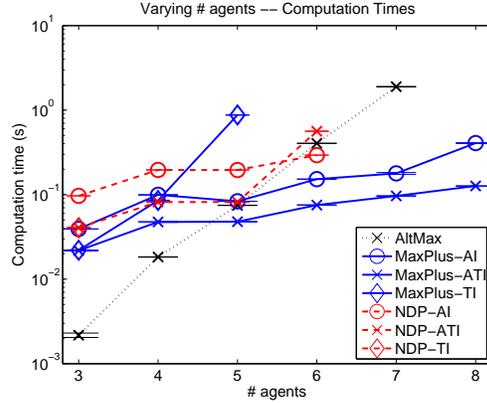
(a) Payoff,  $k = 2, |\Theta_i| = 3, |\mathcal{A}_i| = 3$ .



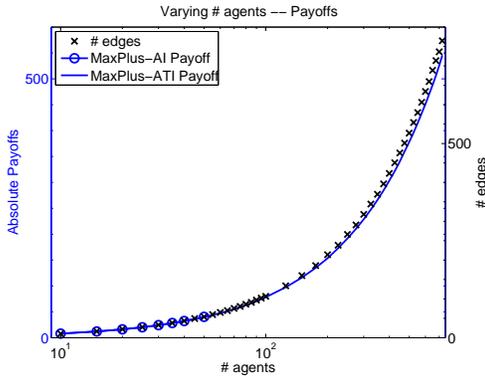
(b) Computation times,  $k = 2, |\Theta_i| = 3, |\mathcal{A}_i| = 3$ .



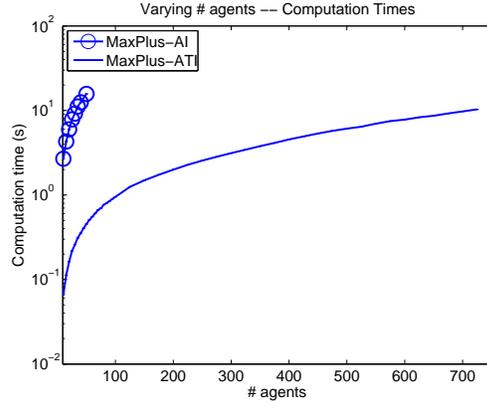
(c) Payoff,  $k = 3, |\Theta_i| = 3, |\mathcal{A}_i| = 3$ .



(d) Computation times,  $k = 3, |\Theta_i| = 3, |\mathcal{A}_i| = 3$ .



(e) Payoff,  $k = 2, |\Theta_i| = 4, |\mathcal{A}_i| = 4$ , and the average number of edges per game (note the two different y-axes indicated with two colors).



(f) Computation times,  $k = 2, |\Theta_i| = 4, |\mathcal{A}_i| = 4$ .

Figure 10: Comparison of the proposed factor-graph methods when scaling the number of agents  $n$ . Plots (a) and (b) consider  $k = 2$  (analogous to Fig. 9a and 9b), while (c) and (d) show results for hyper-edges with  $k = 3$ . Plots (e) and (f) display the scaling behavior for many agents.

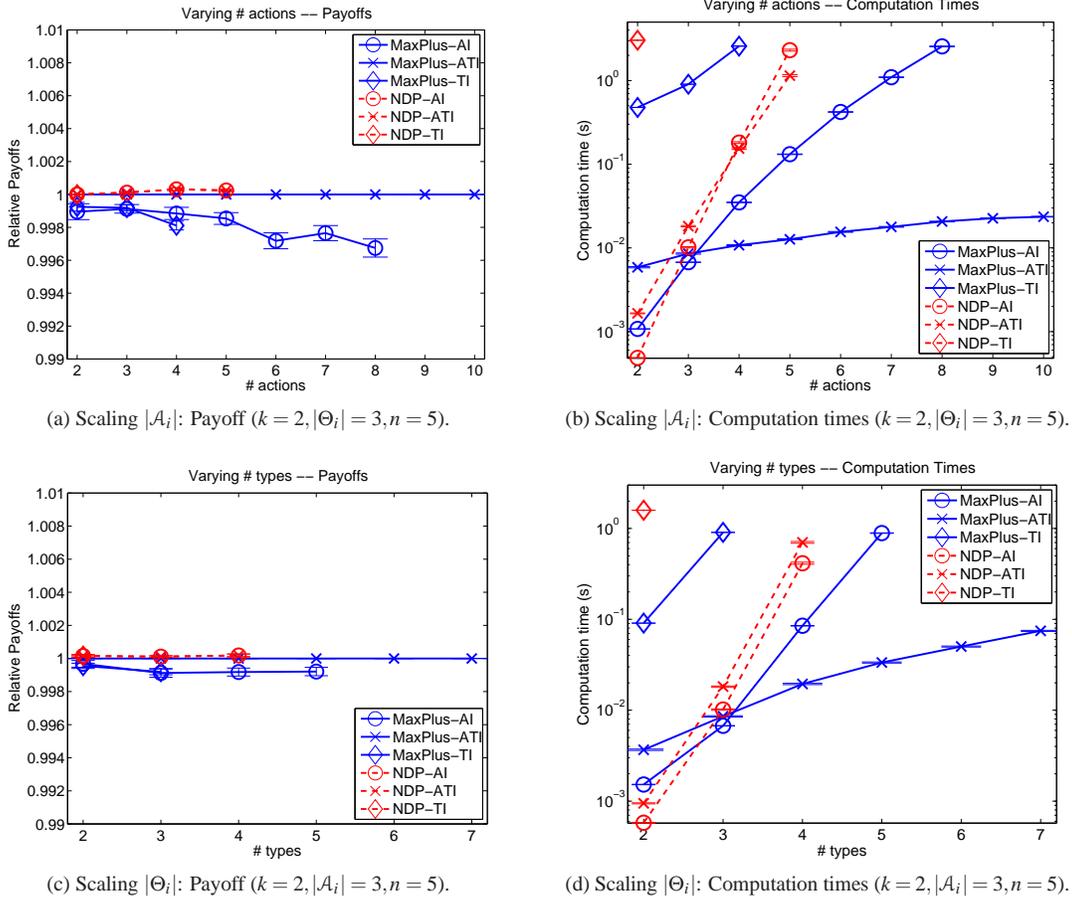


Figure 11: Comparison of the proposed factor-graph methods when scaling the number of actions and the number of types. Plots (a) and (b) consider scaling  $|\mathcal{A}_i|$  (analogous to Fig. 9c and 9d), while (c) and (d) show results increasing  $|\Theta_i|$  (cf. Fig. 9e and 9f).

we apply it to a 2-dimensional implementation of the GENERALIZED FIRE FIGHTING problem described in Sec. 3.2. Each method was limited to 2Gb of memory and allowed 30s computation time.

In this implementation, the  $N_H$  houses are uniformly spread across a 2-dimensional plane, i.e., the  $x$  and  $y$  coordinates for each house are drawn from a uniform distribution over the interval  $[0,1]$ . Similarly, each of the  $n$  agents is assigned a random location and can choose to fight fire at any of  $N_A$  nearest houses, subject to the constraint that at most  $k$  agents can fight fire at a particular house. We enforce this constraint by making a house unavailable to additional agents once it is in the action sets of  $k$  agents.<sup>12</sup> In addition, each agent is assigned, in a similar fashion, the  $N_O$  nearest houses that it can observe. As mentioned in Sec. 3.2, a type  $\theta_i$  is defined by the  $N_O$  observations the agent receives from the surrounding houses:  $\theta_i \in \{F_i, N_i\}^{N_O}$ . We assume that  $N_O \leq N_A$  to ensure that no local payoff function depends on an agent simply due to that agent's type, i.e., an agent never observes a house at which it cannot fight fire.

To ensure there are always enough houses, the number of houses is made proportional to both the number

<sup>12</sup>While this can lead to a sub-optimal assignment, we do not need the best assignment of action sets in order to compare methods.

$x_H$	$\Pr(F x_H)$
0	0.2
1	0.5
> 1	0.8

Table 5: The observation probabilities of a house  $H$ .

of agents and actions:  $N_H = \text{ceil}(N_d \cdot N_A \cdot n)$ , where  $N_d$  is set to 1.2 unless noted otherwise. Each house has a fire level that is drawn uniformly from  $\{0, \dots, N_f - 1\}$ . The probability that an agent receives the observation  $F$  for a house  $H$  depends on its fire level, as shown in Table 5. Observations of different houses by a single agent  $i$  are assumed to be independent, but observations of different agents that can observe the same house are coupled through the hidden state.

The local reward induced by each house is depends on the fire level and the number of agents that chose to fight fire at that house. It is specified by

$$R(x_H, n_{\text{present}}) = -x_H \cdot 0.7^{n_{\text{present}}}.$$

As in Sec. 2.3.1, this reward can be transformed to a (in this case local) utility function by taking the expectation with respect to the hidden state:

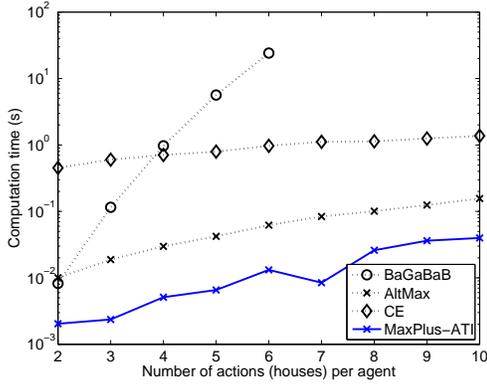
$$u^H(\theta_H, \mathbf{a}_H) = \sum_{x_H=1}^{N_f} \Pr(x_H|\theta_H) R(x_H, \text{CountAgentsAtHouse}(x_H, \mathbf{a}_H)),$$

where `CountAgentsAtHouse()` counts the number of agents for which  $\mathbf{a}_H$  specifies to fight fire at house  $H$ .

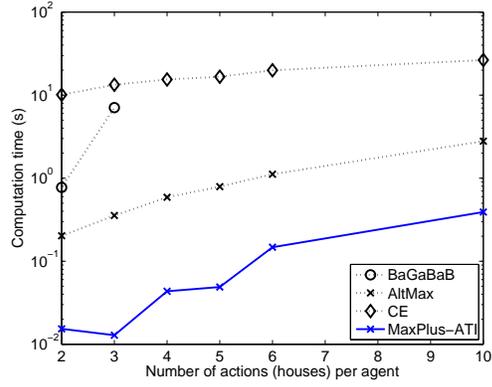
This formulation of SEQUENTIAL FIRE FIGHTING, while still abstract, captures the essential coordination challenges inherent in many realistic problems. For instance, this formulation may directly map to the problem of fire fighting in the Robocup Rescue Simulation league (Kitano et al., 1999): fire fighting agents are distributed in a city and must decide at which houses to fight fire. While limited communication may be possible, it is infeasible for each agent to broadcast all its observations to all the other agents. If instead it is feasible to compute a joint BG-policy based on the agents’ positions, then they can effectively coordinate without broadcasting their observations. When interpreting the houses as queues, it also directly corresponds to problems in queueing networks (Cogill et al., 2004).

The results of our experiments in this domain, shown in Fig. 12, demonstrate that MAX-PLUS has the most desirable scaling behavior in a variety of different parameters. In particular, Fig. 12a and 12b show that all approximate methods scale well with respect to the number of actions per agent, but MAX-PLUS performs best. Fig. 12c shows that this scalability does not come at the expense of solution quality. For all settings, all the methods computed solutions with the same value (other plots of value are thus omitted for brevity). The advantage of exploiting agent independence is illustrated in Fig. 12d, which demonstrates that MAX-PLUS scales well with the number of agents, in contrast to the other methods. In Fig. 12e we varied the  $N_d$  parameter, which determines how many houses are present in the domain. It shows that MAX-PLUS is sensitive to  $k$ , the maximum number of agents that participate in a house. However, it also demonstrates that when the interaction is sparse (i.e., when there are many houses per agent and therefore on average fewer than  $k$  agents per house) the increase in runtime is much better than the worst-case exponential growth.<sup>13</sup> Fig. 12f shows how runtime scales

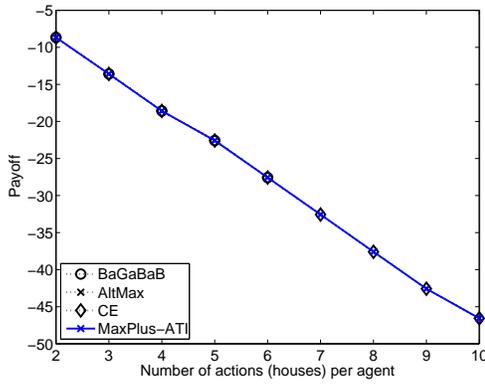
<sup>13</sup>The dense setting  $N_d = 0.5$  does not have data points at  $k = 1$  because in this case there are not enough houses to be assigned to the agents.



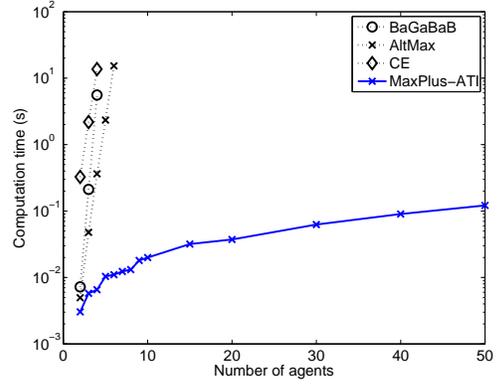
(a) Runtime for varying number of actions when each agent observes 1 house (2 types per agent).



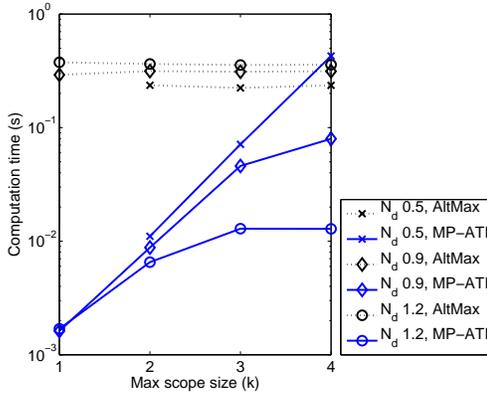
(b) Runtime for varying number of actions when each agent observes 2 houses (4 types per agent).



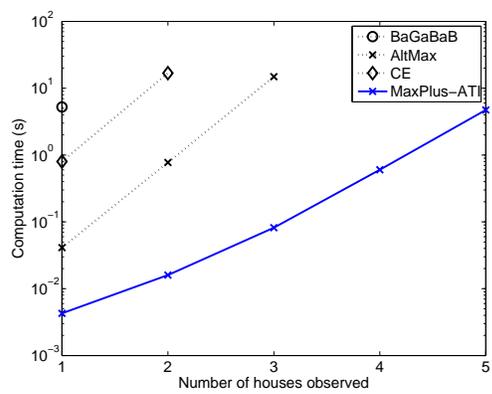
(c) The values for Fig. 12a. The value decreases since the number of houses is proportional to the number of actions.



(d) Varying number of agents. Each agent observes 2 houses (4 types per agent).



(e) Runtime for varying  $k$ , the maximum number of agents that participate in a payoff component. Results are for 4 agents with 3 actions and 2 observed houses (4 types), for different values of  $N_d$ .



(f) Runtime for varying number of houses that are observed per agent. Note that the number of types is exponential in this number. Results are for 4 agents with 5 actions.

Figure 12: Results for the GENERALIZED FIRE FIGHTING problem.

with  $N_O$ , the number of houses observed by each agent. Since the number of types is exponential in  $N_O$ , MAX-PLUS’s runtime is also exponential in  $N_O$ . Nonetheless, it substantially outperforms the other approximate methods.

## 4 Exploiting Independence in Dec-POMDPs

While CBGs model an important class of collaborative decision-making problems, they apply only to *one-shot* settings, i.e., where each agent needs to select only one action. However, the methods for exploiting agent and type independence that we proposed in Sec. 3 can also provide substantial leverage in *sequential* tasks, in which agents take a series of actions over time as the state of the environment evolves. In particular, many sequential collaborative decision-making tasks can be formalized as *decentralized partially observable Markov decision processes (Dec-POMDPs)* (Bernstein et al., 2002). In this section we demonstrate how CBGs can be used in a planning method for the subclass of *factored* Dec-POMDPs with additively factored rewards (Oliehoek et al., 2008c). The resulting method, called FACTORED FSPC, can find approximate solutions for classes of problems that cannot be addressed at all by any other planning methods.

Our aim is not to present FACTORED FSPC as a main contribution of this article. Instead, we merely use it as a vehicle for illustrating the utility of the CGBG framework. Therefore, for the sake of conciseness, we do not describe this method in full technical detail. Instead, we merely sketch the solution approach and supply references to other work containing more detail.

### 4.1 Factored Dec-POMDPs

In a Dec-POMDP, multiple agents must collaborate to maximize the sum of the common rewards they receive over multiple timesteps. Their actions affect not only their immediate rewards but also the state to which they transition. While the current state is not known to the agents, at each timestep each agent receives a private observation correlated with that state. In a factored Dec-POMDP, the state consists of a vector of state variables and the reward function is the sum of a set of local reward functions.

**Definition 4.1.** A factored Dec-POMDP is a tuple  $\langle \mathcal{D}, \mathcal{S}, \mathcal{A}, T, \mathcal{R}, \mathcal{O}, b^0, h \rangle$ , where

- $\mathcal{D} = \{1, \dots, n\}$  is the set of agents.
- $\mathcal{S} = \mathcal{X}_1 \times \dots \times \mathcal{X}_{|\mathcal{X}|}$  is the factored state space. That is,  $\mathcal{S}$  is spanned by  $\mathcal{X} = \{\mathcal{X}_1, \dots, \mathcal{X}_{|\mathcal{X}|}\}$ , a set of state variables, or *factors*. A state corresponds to an assignment of values for all factors  $s = \langle x_1, \dots, x_{|\mathcal{X}|} \rangle$ .
- $\mathcal{A} = \times_i \mathcal{A}_i$  is the set of *joint actions*, where  $\mathcal{A}_i$  is the set of actions available to agent  $i$ .
- $T$  is a transition function specifying the state transition probabilities  $\Pr(s' | s, \mathbf{a})$ .
- $\mathcal{R} = \{R^1, \dots, R^\rho\}$  is the set of  $\rho$  local reward functions. Again, these correspond to a interaction graph with (hyper) edges  $\mathcal{E}$  such that the total immediate reward  $R(s, \mathbf{a}) = \sum_{e \in \mathcal{E}} R^e(\mathbf{x}_e, \mathbf{a}_e)$ .
- $\mathcal{O} = \times_i \mathcal{O}_i$  is the set of joint observations  $\mathbf{o} = \langle o_1, \dots, o_n \rangle$ .
- $O$  is the observation function, which specifies observation probabilities  $\Pr(\mathbf{o} | \mathbf{a}, s')$ .
- $b^0$  is the initial state distribution at time  $t = 0$ .
- $h$  is the horizon, i.e., the number of stages. We consider the case where  $h$  is finite.

At each stage  $t = 0 \dots h - 1$ , each agent takes an individual action and receives an individual observation. Their goal is to maximize the expected *cumulative reward* or *return*. The planning task entails finding a *joint* policy  $\pi = \langle \pi_1, \dots, \pi_n \rangle$ , that specifies an individual policy  $\pi_i$  for each agent  $i$ . Such an individual policy in general specifies an individual action for each action-observation history  $\theta_i^t = (a_i^0, o_i^1, \dots, a_i^{t-1}, o_i^t)$ , e.g.,  $\pi_i(\theta_i^t) = a_i^t$ . However, when only allowing *deterministic* or *pure* policies,  $\pi_i$  maps each observation history  $(o_i^1, \dots, o_i^t) = \bar{o}_i^t \in \bar{O}_i^t$  to an action, e.g.,  $\pi_i(\bar{o}_i^t) = a_i^t$ . In a factored Dec-POMDP, the transition and observation model can be compactly represented in a *dynamic Bayesian network (DBN)* (Boutilier et al., 1999).

#### 4.1.1 Sequential Fire Fighting

As a running example we consider the SEQUENTIAL FIRE FIGHTING problem, which is a sequential variation of GENERALIZED FIRE FIGHTING from Sec. 3.2, originally introduced by Oliehoek et al. (2008c).<sup>14</sup> In this version, each house, instead of simply being on fire or not, has an integer *fire level* that can change over time. Thus, the state of the environment is factored, using one state variable for the fire level in each house. Each agent receives an observation about the house at which it fought fire in the last stage. It observes flames ( $F$ ) at this house with probability 0.2 if  $x_H = 0$ , with probability 0.5 if  $x_H = 1$ , and with probability 0.8 otherwise.

At each stage, each agent  $i$  chooses at which of its assigned houses to fight fire. These actions affect (probabilistically) to what state the environment transitions: the fire level of each house  $H$  is influenced by its previous value, by the actions of the agents that can go to  $H$  and the fire level of the neighboring houses. The transitions in turn determine what reward is generated. Specifically, each house generates a negative reward equal to its expected fire level at the next stage  $x_1^t$ . Thus, the reward function can be described as the sum of local reward functions, one for each house. We consider the case of  $N_H = 4$  as shown in Fig. 4. For house  $H = 1$  the reward is specified by

$$R^1(\mathbf{x}_{\{1,2\}}, a_1) = \sum_{x_1^t} -x_1^t \Pr(x_1^t | \mathbf{x}_{\{1,2\}}, a_1), \quad (4.1)$$

where  $\mathbf{x}_{\{1,2\}}$  denotes  $\langle x_1, x_2 \rangle$ . This formulation is possible because  $x_1, x_2$  and  $a_1$  are the only variables that influence the probability of  $x_1^t$ . Similarly, the other local reward functions are given by  $R^2(\mathbf{x}_{\{1,2,3\}}, \mathbf{a}_{\{1,2\}})$ ,  $R^3(\mathbf{x}_{\{2,3,4\}}, \mathbf{a}_{\{2,3\}})$  and  $R^4(\mathbf{x}_{\{3,4\}}, a_3)$ . For more details about the formulation of SEQUENTIAL FIRE FIGHTING as a factored Dec-POMDP, see (Oliehoek, 2010).

#### 4.1.2 Solving Dec-POMDPs

Solving a Dec-POMDP entails finding an optimal joint policy. Unfortunately, optimally solving Dec-POMDPs is NEXP-complete (Bernstein et al., 2002), as is finding an  $\epsilon$ -approximate solution (Rabinovich et al., 2003).

Given these difficulties, most research efforts have focused on special cases that are more tractable. In particular, assumptions of *transition and observation independence (TOI)* (Becker et al., 2004) have been investigated to exploit independence between agents, e.g., as in ND-POMDPs (Nair et al., 2005; Varakantham et al., 2007). However, given the TOI assumption, many interesting tasks, such as two robots carrying a chair, cannot be modeled. Recently, Witwicki and Durfee (2010) proposed *transition-decoupled* Dec-POMDPs, in which there is limited interaction between the agents. While this approach speeds up both optimal and approximate solutions of this subclass, scalability remains limited (the approach has not been tested with more than two agents) and the sub-class still is quite restrictive (e.g., it does not admit the chair carrying scenario).

Other work has considered approximate methods for the general class of Dec-POMDPs based on representing a Dec-POMDP using CBGs (Emery-Montemerlo et al., 2004) or on approximate dynamic programming

<sup>14</sup>In (Oliehoek et al., 2008c), this problem is referred to as FIREFIGHTINGGRAPH. We use a different name here to distinguish it from GENERALIZED FIRE FIGHTING, which is also graphical.

(Seuken and Zilberstein, 2007). In the remainder of this section, we show CGBGs can help the former category of methods achieve unprecedented scalability with respect to the number of agents.

## 4.2 Factored Dec-POMDPs as Series of CGBGs

A stage  $t$  of a Dec-POMDP can be represented as a CBG, given a *past joint policy*  $\varphi^t$ . Such a  $\varphi^t = (\delta^0, \dots, \delta^{t-1})$  specifies the joint *decision rules* for the first  $t$  stages. An individual decision rule  $\delta_i^t$  of agent  $i$  corresponds to the part of its policy that specifies actions for stage  $t$ . That is,  $\delta_i^t$  maps from observation histories  $\bar{o}_i^t$  to actions  $a_i$ . Given a  $\varphi^t$ , the corresponding CGB is constructed as follows:

- The action sets are the same as in the Dec-POMDP,
- Each  $i$ 's action-observation history  $\bar{\theta}_i^t$  corresponds to its type:  $\theta_i \equiv \bar{\theta}_i^t$ ,
- The probability of joint types is specified by  $\Pr(\bar{\theta}^t | b^0, \varphi^t) = \sum_{s^t} \Pr(s^t, \bar{\theta}^t | b^0, \varphi^t)$ .
- The payoff function  $u(\theta, \mathbf{a}) = Q(\bar{\theta}^t, \mathbf{a})$ , the expected payoff for the remaining stages.

Similarly, a factored Dec-POMDP can be represented by a series of CGBGs. This is possible because, in general, the Q-function is factored. In other words, it is the sum of a set of *local Q-functions* of the following form:  $Q^e(\mathbf{x}_e^t, \bar{\theta}_e^t, \mathbf{a}_e)$ . Each local Q-function depends on a subset of state factors (the state factor scope) and the action-observation histories and actions of a subset of agents (the agent scope) (Oliehoek, 2010).

Constructing a CGBG for a stage  $t$  is similar to constructing a CBG. Given a past joint policy  $\varphi^t$ , we can construct the local payoff functions for the CGBG:

$$u^e(\theta_e, \mathbf{a}_e) \equiv Q_{\varphi^t}^e(\bar{\theta}_e^t, \mathbf{a}_e) = \sum_{\mathbf{x}_e^t} \Pr(\mathbf{x}_e^t | \bar{\theta}_e^t, b^0, \varphi^t) Q^e(\mathbf{x}_e^t, \bar{\theta}_e^t, \mathbf{a}_e). \quad (4.2)$$

As such, the structure of the CGBG is induced by the structure of the Q-value function.

As an example, consider 3-agent  $h = 2$  SEQUENTIAL FIRE FIGHTING. The last stage  $t = 1$  can be represented as a CGBG given a past joint policy  $\varphi^1$ . Also, since it is the last stage, the factored immediate reward function, (e.g., as in Equation 4.1) represents all the expected future reward. That is, it coincides with an optimal factored Q-value function (Oliehoek et al., 2008c) and can be written as follows:

$$Q_{\varphi^1}^1(\bar{\theta}_e^1, \mathbf{a}_e) = \sum_{\mathbf{x}_e^1} \Pr(\mathbf{x}_e^1 | \bar{\theta}_e^1, b^0, \varphi^1) R^1(\mathbf{x}_e^1, \mathbf{a}_e). \quad (4.3)$$

This situation can be represented using a CGBG, by using the Q-value function as the payoff function:  $u^e(\theta_e, \mathbf{a}_e) \equiv Q_{\varphi^1}^e(\bar{\theta}_e^1, \mathbf{a}_e)$  as shown in Fig. 13. The figure shows the 4 components of the CGBG, each one corresponding to the payoff associated with one house. It also indicates an arbitrary BG policy for agent 2. Note that, since components 1 and 4 of the Q-value function have scopes that are ‘subscopes’ of components 2 and 3 respectively, the former can be absorbed into the latter, reducing the number of components without increasing the size of those that remain.

The following theorem by Oliehoek et al. (2008c), shows that modeling a factored Dec-POMDP in this way is in principle exact.

**Theorem 4.1.** *Modeling a factored Dec-POMDP with additive rewards using a series of CGBGs is exact: it yields the optimal solution when using an optimal Q-value function.*

While an optimal Q-value function is factored, the last stage contains the most independence: when moving back in time towards  $t = 0$ , the scope of dependence grows, due to the transition and observation functions. Fig. 14 illustrates this process in SEQUENTIAL FIRE FIGHTING. Thus, even though the value function is factored, the scopes of its components may at earlier stages include all state factors and agents.

$\theta_1$		
(F) $H_1$		-0.25
(F) $H_2$		-1.10
(N) $H_1$		-0.14
(N) $H_2$		-0.79

$Q^{e=1}, \mathbb{A} = \{1\}$

$\theta_1$				
				$\theta_2$
(F) $H_1$				-1.50
(F) $H_2$				-0.50
(N) $H_1$				-1.00
(N) $H_2$				-0.14

$Q^{e=2}, \mathbb{A} = \{1,2\}$

$\theta_3$		
(F) $H_3$		-1.50
(F) $H_4$		-0.51
(N) $H_3$		-1.10
(N) $H_4$		-0.15

$Q^{e=4}, \mathbb{A} = \{3\}$

$\theta_3$				
				$\theta_2$
(F) $H_3$				0
(F) $H_4$				-0.71
(N) $H_3$				-0.58
(N) $H_4$				-1.60

$Q^{e=3}, \mathbb{A} = \{2,3\}$

Figure 13: A CGBG for  $t = 1$  of SEQUENTIAL FIRE FIGHTING. Given a past joint policy  $\varphi^1$ , each joint type  $\theta$  corresponds to a joint action-observation history  $\bar{\theta}^1$ . The entries give the Q-values  $Q^e(\bar{\theta}_e^t, \mathbf{a}_e)$ . Highlighted is an arbitrary policy for agent 2.

### 4.3 Factored Forward Sweep Policy Computation

Defining the payoff function for each CBG representing a stage of a Dec-POMDP requires computing  $Q(\bar{\theta}^t, \mathbf{a})$ , an optimal payoff function. Unfortunately, doing so is intractable. Therefore, Dec-POMDP methods based on CBGs typically use approximate Q-value functions instead. One option ( $Q_{BG}$ ) is to assume that each agent always has access to the joint observations for all previous stages, but can access only its individual observation for the current stage. Another choice is based on the underlying POMDP ( $Q_{POMDP}$ ), i.e., a POMDP with the same transition and observation function in which a single agent receives the joint observations and takes joint actions. A third option is based on the underlying MDP ( $Q_{MDP}$ ), in which this single agent can directly observe the state.

Given an approximate Q-function, an approximate solution for the Dec-POMDP can be computed via *forward-sweep policy computation* (FSPC) by simply solving the CBGs for stages  $0, 1, \dots, h-1$  consecutively. The solution to each CBG is a joint decision rule specified by  $\delta^t \equiv \beta^{t,*}$  that is used to augment the past policy  $\varphi^{t+1} = (\varphi^t, \delta^t)$ . The CBGs are solved consecutively because both the probabilities and the payoffs at each stage depend on the past policy. It is also possible to compute an optimal policy via backtracking, as in *multiagent A\** (MAA\*). However, since doing so is much more computationally intensive, we focus on FSPC in this article.

In the remainder of this section, we describe a method we call FACTORED FSPC for approximately solving a broader class of factored Dec-POMDPs in a way that scales well in the number of agents. The main idea is simply to replace the CBGs used in each stage of the Dec-POMDPs with CGBGs, which are then solved using the methods presented in Sec. 3.

Since finding even bounded approximate solutions for Dec-POMDPs is NEXP-complete, any computationally efficient method must necessarily make unbounded approximations. Below, we describe a set of such approximations designed to make FACTORED FSPC a practical algorithm. While we present only an overview

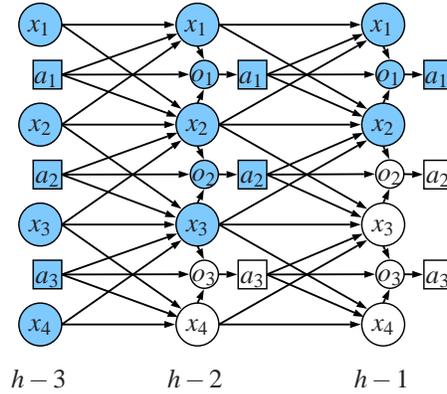


Figure 14: The scope of  $Q^1$ , illustrated by shading, increases when going back in time in SEQUENTIAL FIRE FIGHTING.

here, complete details are available in (Oliehoek, 2010). Also, though the results we present in this article evaluate only the complete resulting method, Oliehoek (2010) empirically evaluated each of the component approximations separately.

#### 4.3.1 Approximate Inference

One source of intractability in FACTORED FSPC lies in the marginalization required to compute the probabilities  $\Pr(\bar{\theta}_e^t | b^0, \varphi^t)$  and  $\Pr(\mathbf{x}_e^t | \bar{\theta}_e^t, b^0, \varphi^t)$ . In particular, constructing each CGBG requires generating each component  $e$  separately. However, as (4.2) shows, in general this requires the probabilities  $\Pr(\mathbf{x}_e^t | \bar{\theta}_e^t, b^0, \varphi^t)$ . Moreover, in any efficient solution algorithm for Dec-POMDPs, the probabilities  $\Pr(\bar{\theta}_e^t | b^0, \varphi^t)$  are necessary, as illustrated by (3.1). Since maintaining and marginalizing over  $\Pr(s, \bar{\theta}^t | b^0, \varphi^t)$  is intractable, we resort to approximate inference, as is standard practice when computing probabilities over states with many factors. Such methods perform well in many cases and the error they introduce can in some cases be theoretically bounded (Boyen and Koller, 1998).

In our case, we use the factored frontier (FF) algorithm (Murphy and Weiss, 2001) to perform approximate inference on a DBN constructed for the past joint policy  $\varphi^t$  under concern. This DBN models stages  $0, \dots, t$  and has both state factors and action-observation histories as its nodes. We use FF because it is simple and allows computation of some useful intermediate representations when a heuristic of the form  $Q^e(\mathbf{x}_e^t, \mathbf{a}_e^t)$  (e.g., factored  $Q_{\text{MDP}}$ ) is used. Other approximate inference algorithms (e.g., Murphy 2002; Mooij 2008a), could also be used.

#### 4.3.2 Approximate Q-Value Functions

Computing the optimal value functions to use as payoffs for the CGBG for each stage is intractable. For small Dec-POMDPs, heuristic payoff functions such as  $Q_{\text{MDP}}$  and  $Q_{\text{POMDP}}$  are typically used instead (Oliehoek et al., 2008b). However, for Dec-POMDPs of the size we consider in this article, solving the underlying MDP or POMDP is also intractable.

Furthermore, factored Dec-POMDPs pose an additional problem: the scopes of  $Q^*$  increase when going backwards in time, such that they are typically fully coupled for earlier stages (see Fig. 14). This problem is exacerbated when  $Q_{\text{MDP}}$  and  $Q_{\text{POMDP}}$  are used as heuristic payoff functions because they become fully coupled through just one backup (due to the maximization over joint actions that is conditioned on the state or belief).

Fortunately, many researchers have considered factored approximations for factored MDPs and factored

POMDPs (Schweitzer and Seidman, 1985; Koller and Parr, 1999, 2000; Schuurmans and Patrascu, 2002; Guestrin et al., 2001a,b, 2003; de Farias and Van Roy, 2003; Szita and Lörincz, 2008). We follow a similar approach for Dec-POMDPs by using value functions with predetermined approximate scopes. The idea is that in many cases the influence of a state factor quickly vanishes with the number of links in the DBN. For instance, in the case of transition and observation independence (TOI), the optimal scopes equal those of the factored immediate reward function. In many cases where there is no complete TOI, the amount of interaction is still limited, making it possible to determine a reduced set of scopes for each stage that affords a good approximate solution. For example, consider the optimal scopes shown in Fig. 14. Though  $x_4$  at  $h - 3$  can influence  $x_2$  at  $h - 1$ , the magnitude of this influence is likely to be small. Therefore, restricting the scope of  $Q^1$  to exclude  $x_4$  at  $h - 3$  is a reasonable approximation.

Following the literature on factored MDPs, we use manually specified scope structures (this is equivalent to specifying basis functions). In the experiments presented in this article, we simply use the immediate reward scopes at each stage, though many alternative strategies are possible. While developing methods for finding such scope structures automatically is an important goal, it is beyond the scope of this article. A heuristic approach suffices to validate the utility of the CGBG framework because our methods require only a good approximate factored value function whose scopes preserve *some* independence.

To compute a heuristic given a specified scope structure, we use an approach we call *transfer planning*. Transfer planning is motivated by the observation that, for a factored Dec-POMDP, the value function is ‘more factored’ than for a factored MDP. In the former, dependence propagates over time, while the latter becomes fully coupled through just one backup. Therefore, it may be preferable to directly approximate the factored Q-value function of the Dec-POMDP rather than the  $Q_{\text{MDP}}$  function. To do so, we use the solution of smaller *source* problems that involve fewer agents. That is, transfer planning directly tries to find heuristic values  $Q_{\phi^e}^e(\hat{\theta}_e^t, \mathbf{a}_e) \equiv Q^s(\hat{\theta}^t, \mathbf{a})$  by solving tasks that are similar (but smaller) and using their value functions  $Q^s$ . The  $Q^s$  can result from the solutions of the smaller Dec-POMDPs, or of their underlying MDP or POMDP. In order to map the values  $Q^s$  of the source tasks to the CGBG components  $Q_{\phi^e}^e$ , we specify a mapping from agents participating in a component  $e$  to agents in the source problem.

Since no formal claims can be made about these approximate Q-values, we cannot guarantee that they constitute an admissible heuristic. However, since we rely on FSPC, which does not backtrack, an admissible heuristic is not necessarily better. Performance depends on the accuracy, not the admissibility of the heuristic (Oliehoek et al., 2008b). The experiments we present below demonstrate that these approximate Q-values are accurate enough to enable high quality solutions.

## 4.4 Experiments

We evaluate FACTORED FSPC on two problem domains: SEQUENTIAL FIRE FIGHTING and the ALOHA problem (Oliehoek, 2010). The latter consists of a number of islands, each equipped with a radio used to transmit messages to its local population. Each island has a queue of messages that it needs to send and at each time step can decide whether or not to send a message. When two neighboring islands attempt to send a message in the same timestep, a collision occurs. Each island can noisily observe whether a successful transmission (by itself or the neighbors), no transmission, or a collision occurred. At each timestep, each island receives a reward of  $-1$  for each message in its queue. ALOHA is considerably more complex than SEQUENTIAL FIRE FIGHTING. First, it has 3 observations per agent, which means that the number of observation histories grows much faster. Also, the transition model of ALOHA is more densely connected than SEQUENTIAL FIRE FIGHTING: the reward component for each island is affected by the island itself and all its neighbors. As a result, in all the ALOHA problems we consider, there is at least one immediate reward function whose scope contains 3 agents, i.e.,  $k = 3$ . Fig. 15 illustrates the case with four islands in a square configuration. The experiments below also consider variants in which islands are connected in a line.

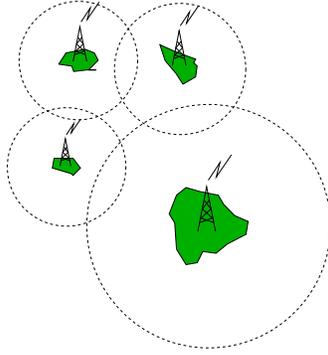


Figure 15: The ALOHA problem with four islands arranged in a square.

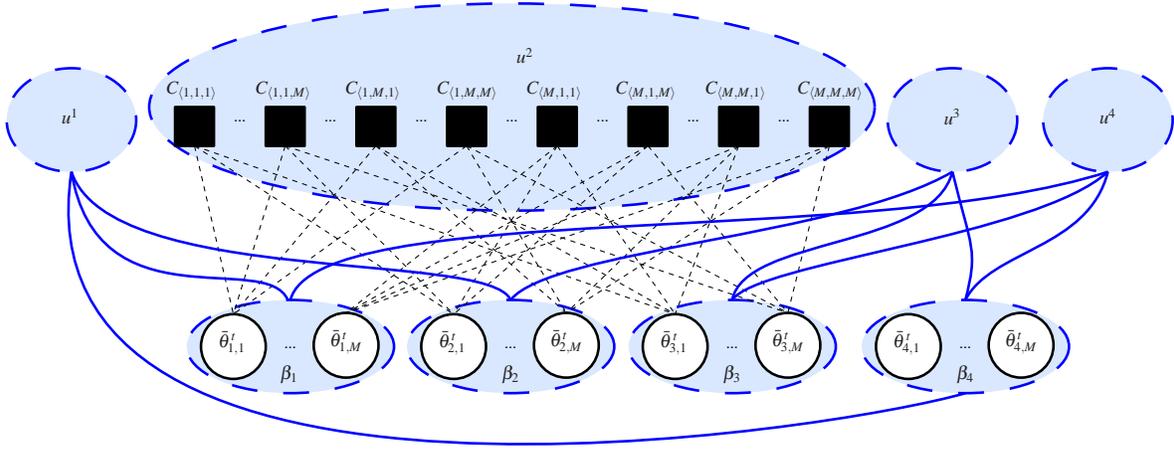


Figure 16: The ATI factor graph for immediate reward scopes in ALOHA. The connections of  $u^2$  are shown in detail by the black dashed lines; connections for other factors are summarized abstractly by the blue solid lines.

In all cases, we use immediate reward scopes that have been reduced (i.e., scopes that form a proper sub-scope of another scope are removed) before computing the factored Q-value functions. This means that for all stages, the factored Q-value function has the same factorization and thus the ATI factor graphs have identical shapes (although the number of types differ). For SEQUENTIAL FIRE FIGHTING, the ATI factor graph for a stage  $t$  is the same as that for GENERALIZED FIRE FIGHTING (see Fig. 6), except that the types  $\theta_i^k$  now correspond to action-observation histories  $\bar{\theta}_{i,k}^t$ . The ATI factor graph for a stage  $t$  of the ALOHA problem is shown in Fig. 16.

To compute  $Q_{TP}$ , the transfer-planning heuristic, for the SEQUENTIAL FIRE FIGHTING problem, we use 2-agent SEQUENTIAL FIRE FIGHTING as the source problem for all the edges and map the lower agent index in a scope to agent 1 and the higher index to agent 2. For the ALOHA problem, we use the 3-island in-line variant as the source problem and perform a similar mapping, i.e., the lowest agent index in a scope is mapped to agent 1, the middle to agent 2 and the highest to agent 3. For both problems we use the  $Q_{MDP}$  and  $Q_{BG}$  heuristic for the source problems.

For problems small enough to solve optimally, we compare the solution quality of FACTORED FSPC to that

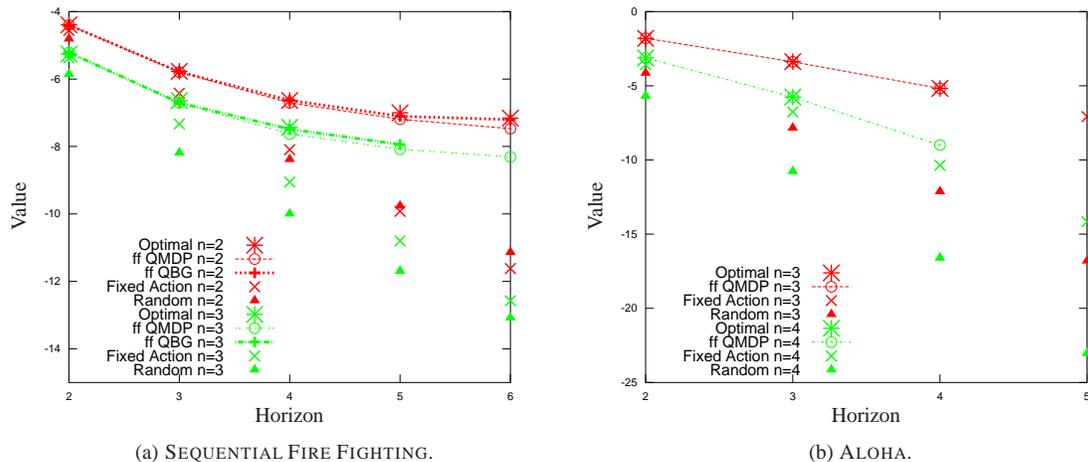


Figure 17: FACTORED FSPC (ff) solution quality compared to optimal and the baselines.

of GMAA\*-ICE, the state-of-the-art method for optimally solving Dec-POMDPs (Spaan et al., 2011). We also compare against several other approximate methods for solving Dec-POMDPs, including non-factored FSPC and direct cross-entropy policy search (DICE) (Oliehoek et al., 2008a), one of the few methods demonstrated to work on Dec-POMDPs with more than three agents that are not transition and observation independent. For non-factored FSPC, we use alternating maximization with 10 restarts to solve the CBGs. For DICE we again use the two parameter settings described in Sec. 3.5 (DICE-normal and DICE-fast).

As baselines, we include a random joint policy and the best joint policy in which each agent selects the same fixed action for all possible histories (though the agents can select different actions from each other). Naturally, these simple policies are suboptimal. However, in the case of the fixed-action baseline, simplicity is a virtue. The reason stems from a fundamental dilemma Dec-POMDP agents face about how much to exploit their private observations. Doing so helps them accrue more local reward but makes their behavior less predictable to other agents, complicating coordination. Because it does not exploit private observations at all, the disadvantages of the fixed-action policy are partially compensated by the advantages of predictability, yielding a surprisingly strong baseline.

We also considered including a baseline in which the agents are allowed to select different actions at each timestep (but are still constrained to a fixed action for all histories of a given length). However, computing the best fixed policy of this form proved intractable.<sup>15</sup> Note that it is not possible to use the solution to the underlying factored MDP as a baseline, for two reasons. First, computing such solutions is not feasible for problems of the size we consider in this article. Second, such solutions would not constitute meaningful baselines for comparison. On the contrary, since such solutions cannot be executed in a decentralized fashion without communication, they provide only a loose upper bound on the performance possible with a Dec-POMDP, as quantitatively demonstrated by Oliehoek et al. (2008b).

The experimental setup is as presented in Sec. 3.5.1, but with a memory limit of 2Gb and a maximum computation time of 1 hour. The reported statistics are means over 10 restarts of each method. Once joint policies have been computed, we perform 10,000 simulation runs to estimate their true values.

<sup>15</sup>In fact, the complexity of doing so is  $O(|A_n^h|)$ , i.e., exponential in both the number of agents and the horizon. This is consistent with the complexity result for the non-observable problem (NP-complete) (Pynadath and Tambe, 2002). By searching for an open-loop plan, we effectively treat the problem as nonobservable.

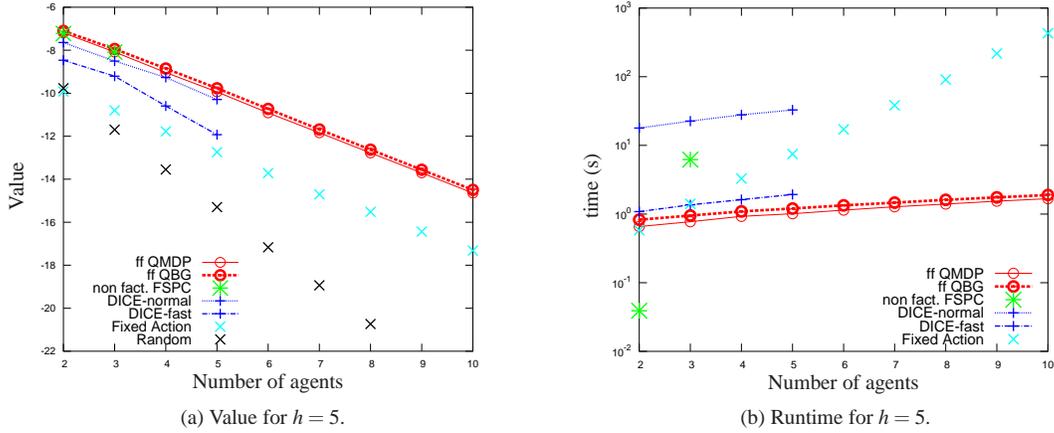


Figure 18: A comparison of FACTORED FSPC (ff) with different heuristics and other methods on the SEQUENTIAL FIRE FIGHTING problem.

Fig. 17 compares FACTORED FSPC’s solutions to optimal solutions on both problems. Fig. 17a show the results for SEQUENTIAL FIRE FIGHTING with two (red) and three agents (green). Optimal solutions were computed up to horizon 6 in the former and horizon 4 in the latter problem. FACTORED FSPC with the  $Q_{BG}$  TP heuristic achieves the optimal value for all these instances. When using the  $Q_{MDP}$  TP heuristic, results are near optimal. For three agents, the optimal value is available only up to  $h = 4$ . Nonetheless, the curve of FACTORED FSPC’s values has the same shape of as that of the optimal values for two agents, which suggests these points are near optimal as well. While the fixed action baselines performs relatively well for shorter horizons, it is worse than random for longer horizons because there always is a chance that the non-selected house will keep burning forever.

Fig. 17b shows results for ALOHA. The  $Q_{BG}$  TP heuristic is omitted since it performed the same as using  $Q_{MDP}$ . For all settings at which we could compute the optimal value, FACTORED FSPC matches this value. Since the ALOHA problem is more complex, FACTORED FSPC has difficulty computing solutions for higher horizons. In addition, the fixed action baseline performs surprisingly well, performing optimally for 3 islands and near optimally for 4 islands. As with SEQUENTIAL FIRE FIGHTING, we expect that it would perform worse for longer horizons: if one agent sends messages for several steps in a row, its neighbor is more likely to have messages backed up in its queue. However, we cannot test this assumption since there are no existing methods capable of solving ALOHA to such horizons against which to compare.

Fig. 18 compares FACTORED FSPC to other approximate methods on the SEQUENTIAL FIRE FIGHTING domain with  $h = 5$ . For all numbers of agents, FACTORED FSPC finds solutions as good as or better than those of non-factored FSPC, DICE-normal, DICE-fast, and the fixed-action and random baselines. In addition, its running time scales much better than that of non-factored FSPC and the fixed-action baseline. Hence, this result highlights the complexity of the problem, as even a simple baseline scales poorly. FACTORED FSPC also runs substantially more quickly than DICE-normal and slightly more quickly than DICE-fast, both of which run out of memory when there are more than five agents.

Fig. 19 presents a similar comparison for the ALOHA problem with  $h = 3$ . DICE-fast is omitted from these plots because DICE-normal outperformed it. Fig. 19a shows that the value achieved by FACTORED FSPC matches or nearly matches that of all the other methods on all island configurations. Fig. 19b shows the runtime

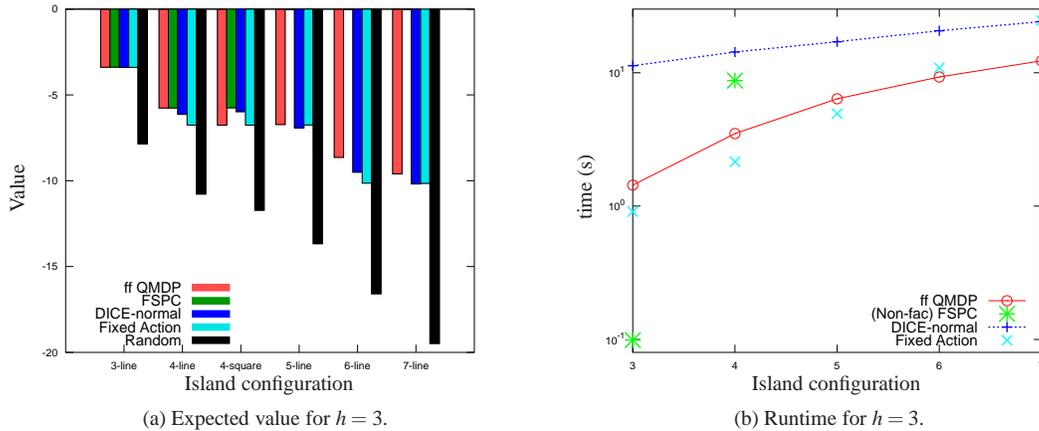


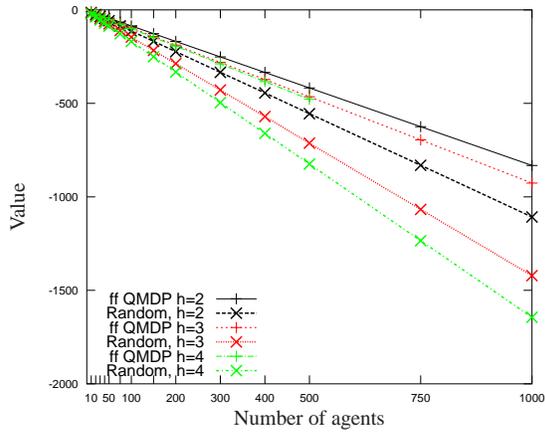
Figure 19: A comparison of FACTORED FSPC with different heuristics and other methods on the ALOHA problem with  $h = 3$ .

results for the inline configurations.<sup>16</sup> While the runtime of FACTORED FSPC is consistently better than that of DICE-normal, non-factored FSPC and the fixed-action baseline are faster for small numbers of agents. Non-factored FSPC is faster for three agents because the problem is fully coupled: there are 3 local payoff functions involving 2, 3, and 2 agents, so  $k=3$ . Thus FACTORED FSPC incurs the overhead of dealing with multiple factors and constructing the FG but achieves no speedup in return. However, the runtime of FACTORED FSPC scales much better as the number of agents increases.

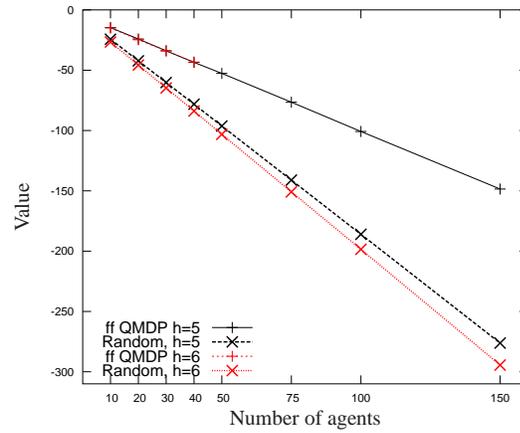
Overall, these results demonstrate that FACTORED FSPC is a substantial improvement over existing approximate Dec-POMDP methods in terms of scaling with respect to the number of agents. However, the ability of FACTORED FSPC to scale with respect to the horizon remains limited, since the number of types in the CGBGs still grows exponentially with the horizon. In future work we hope to address this problem by clustering types (Emery-Montemerlo et al., 2005; Oliehoek et al., 2009; Wu et al., 2011). In particular, by clustering the individual types of an agent that induce similar payoff profiles (Emery-Montemerlo et al., 2005) or probabilities over types of other agents (Oliehoek et al., 2009) it is possible to scale to much longer horizons. When aggressively clustering to a constant number of types, runtime can be made linear in the horizon (Wu et al., 2011). However, since such an improvement is orthogonal to the use of CGBGs, it is beyond the scope of the current article. Moreover, we empirically evaluate the error introduced by a minimal number of approximations required to achieve scalability with respect to the number of agents. Introducing further approximations would confound these results.

Nonetheless, even in its existing form, FACTORED FSPC shows great promise due to its ability to exploit both agent and type independence in the CGBG stage games. To determine the limits of its scalability with respect to the number of agents, we conducted additional experiments applying FACTORED FSPC with the  $Q_{MDP}$  TP heuristic to SEQUENTIAL FIRE FIGHTING with many more agents. The results, shown in Fig. 20, do not include a fixed action baseline because 1) performing simulations of all considered fixed action joint policies becomes expensive for many agents, and 2) the number of such joint policies grows exponentially with the number of agents.

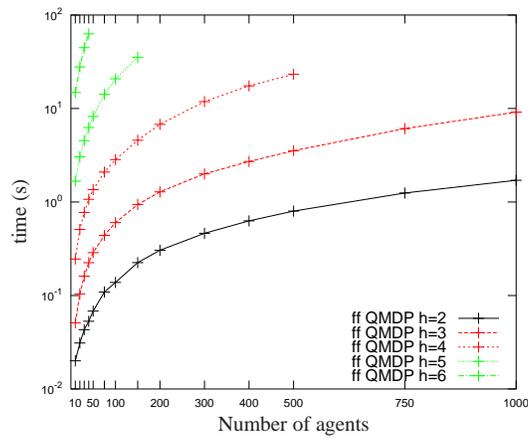
<sup>16</sup>We omit the four agents in a square configuration in order to more clearly illustrate how runtime in the inline configurations scales with respect to the number of agents.



(a) Expected value for  $h = 2, \dots, 4$ .



(b) Expected value for  $h = 5, 6$ .



(c) Runtime for  $h = 2, \dots, 6$ .

Figure 20: FACTORED FSPC results on SEQUENTIAL FIRE FIGHTING with many agents.

As shown in Fig. 20a, FACTORED FSPC successfully computed solutions for up to 1000 agents for  $h = 2, 3$  and 750 agents for  $h = 4$ . For  $h = 5$ , it computed solutions for up to 300 agents; even for  $h = 6$  it computed solutions for 100 agents, as shown in Fig. 20b. Note that, for the computed entries for  $h = 6$ , the expected value is roughly equal to  $h = 5$ . This implies that the probability of any fire remaining at stage  $t = 5$  is close to zero, a pattern we also observed for the optimal solution in Fig. 17. As such, we expect that the found solutions for these settings with many agents are in fact close to optimal. The runtime results, shown in Fig. 20c, increase linearly with respect to the number of agents. While the runtime increases with the number of agents, the bottleneck in our experiments preventing even further scalability was insufficient memory, not computation time.

These results are a large improvement in the state of the art with respect to scalability in the number of agents. Previous approaches for general Dec-POMDPs scaled only to 3 (Oliehoek et al., 2008c) or 5 agents (Oliehoek et al., 2008a).<sup>17</sup> Even when making much stricter assumptions such as transition and observation independence, previous approaches have not scaled beyond 15 agents (Varakantham et al., 2007; Marecki et al., 2008; Varakantham et al., 2009; Kumar and Zilberstein, 2009).

Though these experiments evaluate only the complete FACTORED FSPC method, in (Oliehoek, 2010) we have empirically evaluated each of its component approximations separately. For the sake of brevity, we do not present those experiments in this article. However, the results confirm that each approximation is reasonable. In particular, they show that 1) approximate inference has no significant influence on performance, 2) MAX-PLUS solves CGBGs as well as or better than alternating maximization, 3) the use of 1-step back-projected scopes (i.e., scopes grown by one projection back in the DBN) can sometimes slightly outperform the use of immediate reward scopes, 4) there is not a large performance difference when using optimal scopes, and 5) depending on the heuristic, allowing backtracking can improve performance.

## 5 Related Work

The wide range of research related to the work presented in this article can be mined for various alternative strategies for solving CGBGs. In this section, we briefly survey these alternatives, which fall into two categories: 1) converting CGBGs to other types of games, 2) converting them to constraint optimization problems. We also discuss the relation to the framework of action-graph games (Jiang and Leyton-Brown, 2008).

The first strategy is to convert the CGBG to another type of game. In particular, CGBGs can be converted to CGSGs in the same way CBGs can be converted to SGs. These CGSGs can be modeled using interaction hypergraphs or factor graphs such as those shown in Fig. 2 and solved by applying NDP or MAX-PLUS. However, since this approach does not exploit type independence, the size of local payoff functions scales exponentially in the number of types, making it impractical for large problems. In fact, this approach corresponds directly to the methods, tested in Sec. 3.5, that exploit only agent independence (i.e., NDP-AI and MaxPlus-AI). The poor performance of these methods in those experiments underscores the disadvantages of converting to CGSGs.

CGBGs can also be converted to non-collaborative graphical SGs, for which a host of solution algorithms have recently emerged (Vickrey and Koller, 2002; Ortiz and Kearns, 2003; Daskalakis and Papadimitriou, 2006). However, to do so, the CGBG must first be converted to a CGSG, again forgoing the chance to exploit type independence. Furthermore, in the resulting CGSG, all the payoff functions in which a given agent participates must then be combined into an individual payoff function. This process, which corresponds to converting from an edge-based decomposition to an agent-based one, results in the worst case in yet another exponential increase in the size of the payoff function (Kok and Vlassis, 2006).

Another option is to convert the CGBG into a non-collaborative graphical BG (Singh et al., 2004) by combining the local payoff functions into individual payoff functions directly at the level of the Bayesian game.

<sup>17</sup>However, in very recent work, Wu et al. (2010) present results for up to 20 agents on general Dec-POMDPs.

Again, this may lead to an exponential increase in the size of the payoff functions. Each BNE in the resulting graphical BG corresponds to a local optimum of the original CGBG. Soni et al. (2007) recently proposed a solution method for graphical BGs, which can then be applied to find locally optimal solutions. However, this method converts the graphical BG into a CGSG and thus suffers an exponential blow-up in the size of the payoff functions, just like the other conversion approaches.

The second strategy is to cast the problem of maximization over the CGBG’s factor graph into a (*distributed*) *constraint optimization problem ((D)COP)* (Modi et al., 2005). As such, any algorithm, exact or approximate, for (D)COPs can be used to find a solution for the CGBG (Liu and Sycara, 1995; Yokoo, 2001; Modi et al., 2005; Pearce and Tambe, 2007; Marinescu and Dechter, 2009). Oliehoek et al. (2010) propose a heuristic search algorithm for CBGs that uses this approach and exploits type independence (additivity of the value function) at the level of joint types. Kumar and Zilberstein (2010) employ state-of-the-art methods for weighted constraint satisfaction problems to instances of CBGs in the context of solving Dec-POMDPs. The Dec-POMDPs are solved backwards using dynamic programming, resulting in CBGs with few types but many actions. This approach exploits type independence but has been tested only as part of a Dec-POMDP solution method. Thus, our results in Sec. 4.4 provide additional confirmation of the advantage of exploiting type independence in Dec-POMDPs, while our results in Sec. 3.5 isolate and quantify this advantage in individual CGBGs. Furthermore, the approach presented in this article differs from both these alternatives in that it makes the use of type independence explicit and simultaneously exploits agent independence as well.

Finally, our work is also related to the framework of action-graph games (AGGs), which was recently extended to handle imperfect information and can model any BG (Jiang and Leyton-Brown, 2010). This work proposes two solution methods for general-sum Bayesian AGGs (BAGGs): the Govindan-Wilson algorithm and simplicial subdivision. Both involve computation of the expected payoff of each agent given a current profile as a key step in an inner loop of the algorithm. Jiang and Leyton-Brown (2010) show how this expected payoff can be computed efficiently for each (agent, type)-pair, thereby exploiting type (and possibly agent) independence in this inner loop. As such, this approach may compute a sample Nash equilibrium more efficiently than without using this structure.

On the one hand, BAGGs are more general than CGBGs since they additionally allow representation of context-specific independence and anonymity. Furthermore, the solution method is more general since it works for general-sum games. On the other hand, in the context of collaborative games, a sample Nash equilibrium is not guaranteed to be a PONE (but only a local optimum). In contrast, we solve for the global optimum and thus a PONE. In addition, their approach does not exploit the synergy that independence brings in the identical payoff setting. In contrast, NDP and MAX-PLUS, by operating directly on the factor graph, exploit independence not just within an inner loop but throughout the computation of the solution. Finally, note that our algorithms also work for collaborative BAGGs that possess the same form of structure as CGBGs (i.e., agent and type independence). In cases where there is no anonymity or context-specific independence (e.g., as in the CGBGs generated for Dec-POMDPs), the BAGG framework offers no advantages.

## 6 Future Work

The work presented in this article opens several interesting avenues for future work. A straightforward extension of our methods would replace MAX-PLUS with more recent message passing algorithms for belief propagation that are guaranteed to converge (Globerson and Jaakkola, 2008). Since these algorithms are guaranteed to compute the exact MAP configuration for perfect graphs with binary variables (Jebara, 2009), the resulting approach would be able to efficiently compute optimal solutions for CGBGs with two actions and perfect interaction graphs.

Another avenue would be to investigate whether our algorithms can be extended to work on a broader class of collaborative Bayesian AGGs. Doing so could enable our approach to also exploit context-specific

independence and anonymity. A complementary idea is to extend our algorithms to the non-collaborative case by rephrasing the task of finding a sample Nash equilibrium as one of minimizing regret, as suggested by Vickrey and Koller (2002).

Our approach to solving Dec-POMDPs with CGBGs could be integrated with methods for clustering histories (Emery-Montemerlo et al., 2005; Oliehoek et al., 2009) to allow scaling to larger horizons. In addition, there is great potential for further improvement in the accuracy and efficiency of computing approximate value functions. In particular, the transfer planning approach could be extended to transfer tasks with different action and/or observation spaces, as done in transfer learning (Taylor et al., 2007). Furthermore, it may be possible to automatically identify suitable source tasks and mappings between tasks, e.g., using qualitative DBNs (Liu and Stone, 2006).

## 7 Conclusions

In this article, we considered the interaction of several agents under uncertainty. In particular, we focused on settings in which multiple collaborative agents, each possessing some private information, must coordinate their actions. Such settings can be formalized by the Bayesian game framework. We presented an overview of game-theoretic models used for collaborative decision making and delineated two different types of structure in collaborative games: 1) agent independence, and 2) type independence.

Subsequently, we proposed the *collaborative graphical Bayesian game (CGBG)* as a model that facilitates more efficient decision making by decomposing the global payoff function as the sum of local payoff functions that depend on only a few agents. We showed how CGBGs can be represented as factor graphs (FGs) that capture both agent and type independence. Since a maximizing configuration of the factor graph corresponds to a solution of the CGBG, this representation also makes it possible to effectively exploit this independence.

We considered two solution methods: non-serial dynamic programming (NDP) and MAX-PLUS message passing. The former has a computational complexity that is exponential in the induced tree width of the FG, which we proved to be exponential in the number of individual types. The latter is tractable when there is enough independence between agents; we showed that it is exponential only in  $k$ , the maximum number of agents that participate in the same local payoff function. An empirical evaluation showed that exploiting both agent and type agent independence can lead to a large performance increase, compared to exploiting just one form of independence, without sacrificing solution quality. For example, the experiments showed that this approach allows for the solution of coordination problems with imperfect information for up to 750 agents, limited only by a 1GB memory constraint.

We also showed that CGBGs and their solution methods provide a key missing component in the approximate solution of Dec-POMDPs with many agents. In particular, we proposed FACTORED FSPC, which approximately solves Dec-POMDPs by representing them as a series of CGBGs. To estimate the payoff functions of these CGBGs, we computed approximate factored value functions given predetermined scope structures via a method we call *transfer planning*. It uses value functions for smaller source problems as components of the factored Q-value function for the original target problem. An empirical evaluation showed that FACTORED FSPC significantly outperforms state-of-the-art methods for solving Dec-POMDPs with more than two agents and scales well with respect to the number of agents. In particular, FACTORED FSPC found (near-)optimal solutions on problem instances for which the optimum can be computed. For larger problem instances it found solutions as good as or better than comparison Dec-POMDP methods in almost all cases and in all cases outperformed the baselines. The most salient result from our experimental evaluation is that the proposed method is able to compute solutions for problems that cannot be tackled by any other methods at all (not even the baselines). In particular, it found good solutions for up to 1000 agents, where previously only problems with small to moderate numbers of agents (up to 20) had been tackled.

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## 8 Appendix

**Lemma 8.1** (Complexity of MAX-PLUS). *The complexity of one iteration of MAX-PLUS is*

$$O\left(m^k \cdot k^2 \cdot l \cdot F\right), \quad (8.1)$$

where,  $F$  is the number of factors,  $V$  is the number of variables, the maximum degree of a factor is  $k$ , the maximum degree of a variable is  $l$ , the maximum number of values a variable can take is  $m$ .

*Proof.* We can directly derive that the number of edges is bounded by  $e = F \cdot k$ . Messages sent by a variable are constructed by summing over incoming messages. As a variable has (on average)  $l = \frac{e}{V}$  neighbors, this involves adding (on average)  $l - 1$  incoming messages of size  $m$ . The cost of constructing one message for one variable therefore is:  $O(m \cdot (l - 1))$ . This means that the total cost of constructing all  $e = O(F \cdot k)$  messages sent by variables, one over each edge, is

$$O(m \cdot (l - 1) \cdot F \cdot k) \quad (8.2)$$

Now we consider the messages sent by factors. Recall that the maximum size of a factor is  $m^k$ . The construction of each message entails factor-message addition (see Oliehoek (2010), Sec. 5.5.3) with  $k - 1$  incoming messages, each one has cost  $O(m^k)$ . This leads to a cost of

$$O((k - 1) \cdot m^k) = O(k \cdot m^k)$$

per factor message, and a total cost of

$$O(F \cdot k^2 \cdot m^k). \quad (8.3)$$

The complexity of a single iteration of MAX-PLUS is the sum of (8.2) and (8.3), which can be reduced to (8.1).  $\square$

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